Applications of differentiation

10A Rates of change

The rate of change of a function refers to its gradient. For linear functions the gradient is constant; however, the gradient for other functions such as quadratic or cubic polynomials is continually changing.

The rate of change of position with respect to time is velocity.

The rate of change of velocity with respect to time is acceleration.
Differentiation provides us with a tool to describe the gradient of a function and hence determine its rate of change at any particular point. In essence, while average rates of change can be determined from the original function, differentiation of this function provides a new function that describes the instantaneous rate of change.

Note: The term instantaneous rate of change is often referred to as rate of change.

If \( P(x_1, f(x_1)) \) and \( Q(x_2, f(x_2)) \) are two points on the graph of the function with rule \( y = f(x) \), then the average rate of change of \( y \) with respect to \( x \) over the interval \( x \in [x_1, x_2] \) is equal to the gradient of the straight line \( PQ \).

\[
\text{Average rate of change} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

The instantaneous rate of change finds the rate of change at a specific point. The instantaneous rate of change of \( y(f(x)) \) with respect to \( x \) is given by the derivative \( \frac{dy}{dx} (f'(x)) \).

**WORKED EXAMPLE 1**

If \( f(x) = x^2 - 2x + 4 \), determine:

a. the average rate of change between \( x = 2 \) and \( x = 4 \)

b. a new function that describes the rate of change at any point \( x \)

c. the instantaneous rate of change when \( x = 4 \)

d. parts a, b and c using a calculator.

**WRITE/DISPLAY**

\( a \) \( f(x) = x^2 - 2x + 4 \)

\[
\text{Average rate of change} = \frac{f(4) - f(2)}{4 - 2} = \frac{12 - 4}{2} = 4
\]

\( b \) \( f'(x) = 2x - 2 \)

\( c \) \( f'(4) = 2(4) - 2 = 6 \)

So the rate of change when \( x = 4 \) is 6.

\( d \)

On the Main screen, complete the entry lines as:

Define \( f(x) = x^2 - 2x + 4 \)

\[
\frac{f(4) - f(2)}{4 - 2}
\]

\[
\frac{d}{dx}(f(x))
\]

\[
\frac{d}{dx} (f(x)) \mid x = 4
\]

Press \( \text{Ex} \) after each entry.
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WORKED EXAMPLE 2

A javelin is thrown so that its height, $h$ metres, above the ground is given by the rule: $h(t) = 20t - 5t^2 + 2$, where $t$ represents time in seconds.

a Find the rate of change of the height at any time, $t$.

b Find the rate of change of the height when
i $t = 1$
ii $t = 2$
iii $t = 3$.

c Briefly explain why the rate of change is initially positive, then zero and then negative over the first 3 seconds.

d Find the rate of change of the height when the javelin first reaches a height of 17 metres.

THINK

a 1 Write the rule.
2 Differentiate $h(t)$.

b 1 Evaluate $h'(1)$.
2 Evaluate $h'(2)$.
3 Evaluate $h'(3)$.

c For rates of change:
Positive means increasing.
Zero means neither increasing nor decreasing.
Negative means decreasing.

d 1 Find the time at which the javelin is 17 m above the ground, by substituting $h = 17$ into $h(t)$.
2 Make RHS = 0.
3 Divide both sides by $-5$.
4 Factorise then solve for $t$.
   Note: The quadratic formula could also be used to solve for $t$.
5 The first time it reaches 17 m is the smaller value of $t$.
6 Evaluate $h'(1)$.

WRITE

a $h(t) = 20t - 5t^2 + 2$
   $h'(t) = 20 - 10t$

b i $h'(1) = 20 - 10(1)$
   = 10 m/s
ii $h'(2) = 20 - 10(2)$
   = 0 m/s
iii $h'(3) = 20 - 10(3)$
   = -10 m/s

c The javelin travels upwards during the first 2 seconds.
When $t = 2$ seconds, the javelin has reached its maximum height.
When $t > 2$ seconds, the javelin is travelling downwards.

d $20t - 5t^2 + 2 = 17$
   $5t^2 + 20t - 15 = 0$
   $t^2 - 4t + 3 = 0$
   $(t - 1)(t - 3) = 0$
   $t = 1$ or 3

The javelin first reaches 17 m when $t = 1$ s.

$h'(1) = 20 - 10(1)$
   = 10 m/s
Rate of change of height is 10 m/s.
It is worth noting that there are two common ways of writing the derivative as a function. For example, the derivative of the function \( P(x) = x^2 + 5x - 7 \) may be written as \( P'(x) = 2x + 5 \) or as \( \frac{dP}{dx} = 2x + 5 \).

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**WORKED EXAMPLE 3**

The shockwave from a nuclear blast spreads out at ground level in a circular manner.

- **a** Write down a relationship between the area of ground, \( A \) km\(^2\), over which the shockwave passes and its radius, \( r \) km.
- **b** Find the rate of change of \( A \) with respect to \( r \).
- **c** Find the rate of change of \( A \) when the radius is 2 km.
- **d** What is the rate of change of \( A \) when the area covered is 314 km\(^2\)?

**THINK**

- **a** State the formula for the area of a circle.
- **b** Differentiate \( A(r) \).
- **c** Substitute \( r = 2 \) into \( A'(r) \).
  
  *Note: The units for the rate of change of \( A \) (km\(^2\)) with respect to \( r \) (km) are km\(^2\) per km or km\(^2\)/km.*

- **d** 1. Substitute \( A = 314 \) into the area function \( A(r) \) and solve for \( r \).
  2. Find the rate of change when \( r = 10 \).

**WRITE**

- **a** \( A(r) = \pi r^2 \)
- **b** \( A'(r) = 2\pi r \)
- **c** \( A'(2) = 2\pi(2) = 12.57 \)
  
  Rate of change of \( A \) when the radius is 2 km is 12.57 km\(^2\)/km.

- **d** \( A(r) = \pi r^2 \)
  
  \[
  r^2 = \frac{314}{\pi} = 99.95 \quad r = 10 \text{ since } r > 0
  \]

  Rate of change of \( A \) when area is 314 km\(^2\) is 62.83 km\(^2\)/km.

**REMEMBER**

The rate of change of a function refers to its gradient.

Average rate of change = \( \frac{\text{change in } f(x)}{\text{change in } x} \).

Average rates of change are calculated using the original function.

Differentiation of this function is needed in order to calculate instantaneous rates of change at specific points.
1. If \( f(x) = x^2 + 5x + 15 \), find:
   a. the average rate of change between \( x = 3 \) and \( x = 5 \)
   b. a new function that describes the rate of change
   c. the instantaneous rate of change when \( x = 5 \).

2. A balloon is inflated so that its volume, \( V \) cm\(^3\), at any time, \( t \) seconds, later is:
   \[ V = \frac{-8}{5}t^3 + 24t^2, \quad t \in [0, 10] \]
   a. What is the volume of the balloon when:
      i. \( t = 0 \)?
      ii. \( t = 10 \)?
   b. Hence, find the average rate of change between \( t = 0 \) and \( t = 10 \).
   c. Find the rate of change of volume when:
      i. \( t = 0 \)
      ii. \( t = 5 \)
      iii. \( t = 10 \).

3. The average rate of change between \( x = 1 \) and \( x = 3 \) for the function \( y = x^2 + 3x + 5 \) is:
   A. 1
   B. 9
   C. 5
   D. 3
   E. 7

4. The instantaneous rate of change of the function \( f(x) = x^3 - 3x^2 + 4x \), when \( x = -2 \) is:
   A. 2
   B. -2
   C. 28
   D. 3
   E. 12

5. If the rate of change of a function is described by \( \frac{dy}{dx} = 2x^2 - 7x \), then the function could be:
   A. \( y = 6x^3 - 14x \)
   B. \( y = \frac{2}{3}x^3 - 7x \)
   C. \( y = \frac{2}{3}x^3 - \frac{7}{2}x^2 + 5 \)
   D. \( y = x^3 - \frac{3}{2}x^2 + 2 \)
   E. \( 2x^2 - 7x + 5 \)

6. In a baseball game the ball is hit so that its height above the ground, \( h \) metres, is \( h(t) = 1 + 18t - 3t^2 \) t seconds after being struck.
   a. Find the rate of change, \( h'(t) \).
   b. Calculate the rate of change of height after:
      i. 2 seconds
      ii. 3 seconds
      iii. 4 seconds.
   c. What happens when \( t = 3 \) seconds?
   d. Find the rate of change of height when the ball first reaches a height of 16 metres.

7. The position, \( x \) metres, of a lift (above ground level) at any time, \( t \) seconds, is given by:
   \( x(t) = -2t^2 + 40t \)
   a. Find the rate of change of displacement (velocity) at any time, \( t \).
   b. Find the rate of change when:
      i. \( t = 5 \)
      ii. \( t = 9 \)
      iii. \( t = 11 \).
   c. What happened between \( t = 9 \) and \( t = 11 \)?
   d. When and where is the rate of change zero?

8. The number of seats, \( N \), occupied in a soccer stadium \( t \) hours after the gates are opened is given by:
   \( N = 500t^2 + 3500t, \quad t \in [0, 5] \)
   a. Find \( N \) when:
      i. \( t = 1 \)
      ii. \( t = 3 \).
   b. What is the average rate of change between \( t = 1 \) and \( t = 3 \)?
   c. Find the instantaneous rate when:
      i. \( t = 0 \)
      ii. \( t = 1 \)
      iii. \( t = 3 \)
      iv. \( t = 4 \).
   d. Why is the rate increasing in the first 4 hours?
The weight, \( W \) kg, of a foal at any time, \( t \) weeks, after birth is given by:

\[
W = 80 + 12t - \frac{3}{10}t^2 \quad \text{where} \quad 0 \leq t \leq 20
\]

9 a What is the weight of the foal at birth?
b Find an expression for the rate of change of weight at any time, \( t \).
c Find the rate of change after:
   i 5 weeks
   ii 10 weeks
   iii 15 weeks.
d Is the rate of change of the foal’s weight increasing or decreasing?
e When does the foal weigh 200 kg?

10 The weekly profit, \( P \) (hundreds of dollars), of a factory is given by \( P = 4.5n - n^2 \), where \( n \) is the number of employees.

a Determine \( \frac{dP}{dn} \).
b Hence, find the rate of change of profit, in dollars per employee, if the number of employees is:

   i 4
   ii 16
   iii 25.
c Find \( n \) when the rate of change is zero.

11 Gas is escaping from a cylinder so that its volume, \( V \) cm\(^3\), \( t \) seconds after the leak starts, is described by \( V = 2000 - 20t - \frac{1}{100}t^2 \).

a Evaluate the rate of change after:

   i 10 seconds
   ii 50 seconds
   iii 100 seconds.
b Is the rate of change ever positive? Why?

12 Assume an oil spill from an oil tanker is circular and remains that way.

a Write down a relationship between the area of the spill, \( A \) m\(^2\), and the radius, \( r \) metres.
b Find the rate of change of \( A \) with respect to the radius, \( r \).
c Find the rate of change of \( A \) when the radius is:

   i 10 m
   ii 50 m
   iii 100 m.
d Is the area increasing more rapidly as the radius increases? Why?

13 A spherical balloon is being inflated.

a Express the volume of the balloon, \( V \) m\(^3\), as a function of the radius, \( r \) metres.
b Find the rate of change of \( V \) with respect to \( r \).
c Find the rate of change when the radius is:

   i 0.1 m
   ii 0.2 m
   iii 0.3 m.

14 A rectangular fish tank has a square base, with its height being equal to half its base length.

a Express the length and width of the base in terms of its height, \( h \).
b Hence, express the volume, \( V \) m\(^3\), in terms of the height, \( h \), only.
c Find the rate of change of \( V \) when:

   i \( h = 1 \) m
   ii \( h = 2 \) m
   iii \( h = 3 \) m.

15 For the triangular package shown, find:

a \( x \) in terms of \( h \)
b the volume, \( V \), as a function of \( h \) only
c the rate of change of \( V \) when:

   i \( h = 0.5 \) m
   ii \( h = 1 \) m.
16 A new estate is to be established on the side of a hill.

Regulations will not allow houses to be built on slopes where the gradient is greater than 0.45. If the equation of the cross-section of the hill is

\[ y = -0.000\ 02x^3 + 0.006x^2 \]

find:

a the gradient of the slope \( \frac{dy}{dx} \)

b the gradient of the slope when \( x \) equals:
   i 160  
   ii 100  
   iii 40  
   iv 20  

c the values of \( x \) where the gradient is 0.45

d the range of heights for which houses cannot be built on the hill.

17 A bushfire burns out \( A \) hectares of land, \( t \) hours after it started, according to the rule:

\[ A = 90t^2 - 3t^3 \]

a At what rate, in hectares per hour, is the fire spreading at any time, \( t \)?

b What is the rate when \( t \) equals:
   i 0? 
   ii 4? 
   iii 8? 
   iv 10? 
   v 12? 
   vi 16? 
   vii 20?

c Briefly explain how the rate of burning changes during the first 20 hours.

d Why isn’t there a negative rate of change in the first 20 hours?

10B Sketching graphs containing stationary points

The derivative of a function gives its gradient function — that is, it gives the gradient of a tangent to the curve for any specified value of the independent variable. When the derivative equals zero, the tangent is horizontal. The point, or points, on the curve where this occurs are called stationary points.

In other words, a function \( f(x) \) has stationary points when \( f'(x) = 0 \).

Stationary points can take the form of:

1. a local minimum turning point
2. a local maximum turning point
3. a stationary point of inflection.

**Local minimum turning point**

Just to the left of \( a \), the gradient is negative; that is, if \( x < a \), but close to \( a \), then \( f'(x) < 0 \).

At the point where \( x = a \) the gradient is zero; that is, at \( x = a \), \( f'(x) = 0 \).

Just to the right of \( a \) the gradient is positive; that is, if \( x > a \), but close to \( a \), \( f'(x) > 0 \).
In other words, for a stationary point at \( x = a \), if the gradient changes from negative to positive as we move from left to right in the vicinity of \( a \), it is a local minimum.

**Local maximum turning point**

At \( x < a \), but close to \( a \), \( f'(x) > 0 \).

At \( x = a \), \( f'(x) = 0 \).

If \( x > a \), but close to \( a \), \( f'(x) < 0 \).

In other words, for a stationary point at \( x = a \), if the gradient changes from positive to negative as we move from left to right in the vicinity of \( a \), it is a local maximum.

The term *local maximum* or *local minimum* implies that the function has a maximum or minimum in the vicinity of \( x = a \). This is important because some functions can have more than one stationary point.

**Stationary point of inflection**

At \( x < a \), but close to \( a \), \( f'(x) > 0 \).

At \( x = a \), \( f'(x) = 0 \).

At \( x > a \), but close to \( a \), \( f'(x) < 0 \).

In other words, for a stationary point at \( x = a \), if the gradient remains positive or negative in the vicinity of \( a \), it is a stationary point of inflection.

**WORKED EXAMPLE 4**

If \( f(x) = x^3 - 6x^2 - 15x \), find:

a the value(s) of \( x \) where the gradient is zero

b the stationary point(s).

**WRITE**

a \( f(x) = x^3 - 6x^2 - 15x \)

\( f'(x) = 3x^2 - 12x - 15 \)

For stationary points: \( f'(x) = 0 \)

\( 3x^2 - 12x - 15 = 0 \)

\( 3(x^2 - 4x - 5) = 0 \)

\( x^2 - 4x - 5 = 0 \)

\( (x - 5)(x + 1) = 0 \)

\( x = 5 \) or \( x = -1 \)
b Substitute each value of \( x \) into \( f(x) \) to find the corresponding \( y \)-values.

\[
b f(5) = (5)^3 - 6(5)^2 - 15(5) = -100 \\
b f(-1) = (-1)^3 - 6(-1)^2 - 15(-1) = 8 \\
\]

Stationary points occur at \((5, -100)\) and \((-1, 8)\).

**Worked Example 5**

Sketch the graph of the function \( f(x) = 5 + 4x - x^2 \), labelling all intercepts and stationary points.

**Think**

1. Write the function.
2. Find the \( y \)-intercept by letting \( x \) equal 0.
3. Find the \( x \)-intercepts by letting \( f(x) = 0 \).
4. Differentiate \( f(x) \) to find the gradient function \( f'(x) \).
5. Solve \( f'(x) = 0 \) to find the \( x \)-value(s) of each stationary point.
6. Substitute this value of \( x \) into \( f(x) = 5 + 4x - x^2 \) to find the corresponding \( y \)-value.
7. Determine the nature of the stationary point at \( x = 2 \) by evaluating \( f''(x) \) to the left and right, say at \( x = 1 \) and \( x = 3 \).
8. Complete a gradient table. Since the gradient changes from positive to negative as we move from left to right in the vicinity of \( x = 2 \), the stationary point \((2, 9)\) is a local maximum.

**Write/Draw**

\[
f(x) = 5 + 4x - x^2 \\
\text{y-intercept: } x = 0, \\
f(0) = 5 + 4(0) - (0)^2 = 5 \\
\text{so y-intercept is } (0, 5). \\
\]

\[
x-intercepts: \text{if } f(x) = 0, \\
5 + 4x - x^2 = 0 \\
x^2 - 4x - 5 = 0 \\
(x + 1)(x - 5) = 0 \\
x = -1 \text{ or } x = 5 \\
\text{so x-intercepts are } (-1, 0) \text{ and } (5, 0). \\
\]

\[
f(x) = 5 + 4x - x^2 \\
f'(x) = 4 - 2x \\
\]

For stationary points: \( f'(x) = 0 \)
\[
4 - 2x = 0 \\
-2x = -4 \\
x = 2 \\
\]

\[
f(2) = 5 + 4(2) - (2)^2 = 9 \\
\text{so there is a stationary point at } (2, 9). \\
\]

\[
x < 2: f''(1) = 4 - 2(1) = 2 \\
\therefore f'(x) > 0 \\
x > 2: f''(3) = 4 - 2(3) = -2 \\
\therefore f'(x) < 0 \\
\]

\[
\begin{array}{|c|c|c|c|}
\hline
x & 1 & 2 & 3 \\
\hline
\text{Sign of } f'(x) & + & 0 & - \\
\hline
\text{Slope} & \\
\hline
\end{array}
\]

(2, 9) is a local maximum.
Sketch the graph.

\[ y = f(x) \]

\[ (2, 9) \]

Find the stationary points and determine their nature for the function

\[ f(x) = x^3 - x^2 - 8x + 8. \]

b Find the coordinates of all intercepts.

c Sketch the graph of \( f(x) \) showing all stationary points and intercepts.

**WORKED EXAMPLE 6**

**THINK**

a 1 Write the rule for \( f(x) \).

2 Differentiate \( f(x) \) to find \( f'(x) \).

3 Solve \( f'(x) = 0 \) to find the \( x \)-values of each stationary point.

4 Substitute each value of \( x \) into \( f(x) \) to find the \( y \)-coordinates of the stationary points.

5 Alternatively, all of this working can be done on a CAS calculator.

On the Main screen, complete the entry lines as:
Define \( f(x) = x^3 - x^2 - 8x + 8 \)

\[ \text{solve} \left\{ \frac{d}{dx}(f(x)) = 0, x \right\} \]

\( f \left\{ \left\{ -\frac{4}{3}, 2 \right\} \right\} \)

Press \( \text{ENTRY} \) after each entry.

**WRITE/DISPLAY**

a \( f(x) = x^3 - x^2 - 8x + 8 \)

\[ f'(x) = 3x^2 - 2x + 8 \]

For stationary points, solve \( f'(x) = 0 \) for \( x \).

\[ 3x^2 - 2x + 8 = 0 \]

\[ (3x + 4)(x - 2) = 0 \]

\[ x = -\frac{4}{3} \text{ or } x = 2 \]

\[ f\left( -\frac{4}{3} \right) = \left( -\frac{4}{3} \right)^3 - \left( -\frac{4}{3} \right)^2 - 8\left( -\frac{4}{3} \right) + 8 = \frac{392}{27} = 14 \frac{44}{27} \]

\[ \therefore \left( -\frac{4}{3}, 14 \frac{44}{27} \right) \text{ is one stationary point.} \]

\[ f(2) = (2)^3 - (2)^2 - 8(2) + 8 = -4 \]

\[ \therefore (2, -4) \text{ is another stationary point.} \]

6 Determine the nature of the stationary point at \( x = -\frac{4}{3} \) by evaluating \( f'(x) \) to the left and right. Choose \( x = -2 \) and \( x = -1 \).

\[ x < -\frac{4}{3}: f'(-2) = 3(-2)^2 - 2(-2) - 8 = 8 \]

\[ \therefore f'(x) > 0 \]

\[ x > -\frac{4}{3}: f'(-1) = 3(-1)^2 - 2(-1) - 8 = -3 \]

\[ \therefore f'(x) < 0 \]
Complete a gradient table and state the type of stationary point. Since the gradient changes from positive to negative as we move from left to right in the vicinity of \(x = -\frac{4}{3}\), the stationary point \((-\frac{4}{3}, \frac{14}{27})\) is a local maximum.

Determine the nature of the stationary point at \(x = 2\) by evaluating \(f'(x)\) to the left and right. Choose \(x = 1\) and \(x = 3\).

To find the \(x\)-intercepts, factorise \(f(x)\) by long division, or use another appropriate method.

Solve \(f(x) = 0\) for \(x\)-intercepts.

Evaluate \(f(0)\) to find the \(y\)-intercept.

Sketch the graph of \(f(x)\) showing all stationary points and axes intercepts.

**Worked Example 7**

The curve with equation \(y = ax^2 + bx + 7\) has a stationary point at \((-2, 10)\). Find the values of \(a\) and \(b\).

1. Write the rule.
2. Differentiate \(y\) to find the gradient function.

\[
\begin{align*}
\frac{dy}{dx} &= 2ax + b \\
\end{align*}
\]
3. Put \( \frac{dy}{dx} = 0 \) and substitute \( x = -2 \) into \( \frac{dy}{dx} \), as \( x = -2 \) is a stationary point.

4. Substitute \( x = -2 \) into \( y \) and put \( y = 10 \) to get another equation with \( a \) and \( b \).

5. Solve the simultaneous equations [1] and [2]. Add equations [1] and [2] to eliminate \( a \) and solve for \( b \).

6. Substitute \( b = -3 \) into equation [1] to find \( a \).

7. Write the values of \( a \) and \( b \).

8. Alternatively, all of this working can be done on a CAS calculator. On the Main screen, complete the entry line as:
   Define \( f(x) = ax^2 + bx + 7 \)
   Then press \( \text{Solv.} \)
   To set up the equation to be solved, tap:
   • \( 2\text{D} \)
   • \( \text{Mat} \)
   Enter the equations as shown on the right. Press \( \text{Solv.} \).

### REMEMBER

When sketching graphs of polynomial functions, four main features should be indicated:
1. the y-intercept (found by calculating \( y \) when \( x = 0 \))
2. the x-intercept(s) (found by solving the equation for \( x \) when \( y = 0 \))
3. the stationary point(s) (found by solving the equation \( \frac{dy}{dx} = 0 \))
4. the type of stationary point(s) (found by checking the sign of the gradient to the left and right of the stationary point).

### EXERCISE 10B

**Sketching graphs containing stationary points**

1. **WE4a** For each of the following functions determine the value(s) of \( x \) where the gradient is zero.
   - \( a \) \( f(x) = x^2 + 2x \)
   - \( b \) \( f(x) = x^2 - 8x + 5 \)
   - \( c \) \( f(x) = x^3 - 3 x^2 \)
   - \( d \) \( f(x) = 2x^3 + 6x^2 - 18x + 1 \)
   - \( e \) \( y = (x+6)(x-2) \)
   - \( f \) \( y = x^2(x-1) \)
   - \( g \) \( y = 10 + 4x - x^2 \)
   - \( h \) \( y = \frac{1}{3} x^3 - 3x^2 + 5x - 2 \)

2. **WE4b** For each function in question 1 determine all of the stationary points.
3. If \( f(x) = x^2 - 8x + 1 \):
   a. show that there is a stationary point when \( x = 4 \)
   b. evaluate \( f'(3) \) and \( f'(5) \)
   c. state which type of stationary point it is.

4. For the function \( f(x) = 5 - x^2 \):
   a. find \( x \) when \( f(x) = 0 \)
   b. state which type of stationary point it has.

5. If \( f(x) = x^3 - 4 \) then:
   a. show that there is a stationary point when \( x = 0 \) only
   b. find \( f'(-1) \) and \( f'(1) \)
   c. state which type of stationary point it is.

6. If \( f(x) = \frac{1}{3}x^3 - x^2 - 3x + 5 \):
   a. show there are stationary points when \( x = -1 \) and \( x = 3 \)
   b. evaluate \( f'(-2), f'(0) \) and \( f'(4) \)
   c. state which type of stationary points they are.

7. MC When \( x = 1 \), the curve \( y = 2x^2 - 3x + 1 \):
   A. is decreasing
   B. has a local maximum
   C. has a local minimum
   D. is increasing
   E. does not exist

8. MC When \( x = -1 \) the function \( y = x^3 - 2x^2 - 7x \):
   A. is decreasing
   B. has a local maximum
   C. has a local minimum
   D. is increasing
   E. does not exist

9. MC The graph below that best represents a function with \( f'(-2) = 0 \), \( f'(x) < 0 \) if \( x < -2 \),
   and \( f'(x) > 0 \) if \( x > -2 \) is:

   [Graph options A, B, C, D, E]
10 MC \( f'(1) = f'(4) = 0 \) and \( f'(x) < 0 \) if \( 1 < x < 4 \) and \( f'(x) > 0 \) if \( x < 1 \) and \( x > 4 \). The graph below that satisfies these conditions is:

A [Graph A]

B [Graph B]

C [Graph C]

D [Graph D]

E [Graph E]

11 WES For each of the following, find the stationary points and determine their nature.

a) \( y = x^2 + 6x + 2 \)

b) \( y = 8x - 2x^2 \)

c) \( y = x^3 - x^2 \)

d) \( y = x^3 + \frac{1}{2}x^2 - 3 \)

e) \( y = \frac{3}{5}x^3 - \frac{1}{2}x^2 - 2x \)

f) \( y = (x - 1)^3 \)

g) \( y = x^3 + 3 \)

h) \( y = x^3 - 27x + 5 \)

12 WES Use a CAS calculator to help sketch the graphs of the following functions, labelling all intercepts and stationary points.

a) \( f(x) = x^2 - 2x - 3 \)

b) \( f(x) = x^3 - 3x - 2 \)

c) \( f(x) = x^3 - 2x^2 + x \)

d) \( f(x) = x^3(3 - x) \)

e) \( f(x) = x^3 + 4x^2 + 4x \)

f) \( f(x) = x^3 - 4x^2 - 11x + 30 \)

i) \( f(x) = 24 + 10x - 3x^3 - x^3 \)

j) \( f(x) = 8 - x^3 \)

k) \( f(x) = 2x^2 + bx + 8 \) has a turning point at \( x = 3 \). Find the value of \( b \).

15 WES For the functions \( y = -2x^2 - 5x + 8 \) and \( y = 2x^3 - x^2 - 4x + 5 \):

a) use a CAS calculator to sketch the graph of each function showing all coordinates of stationary points

b) hence, find the \( x \)-values for which:

i) \( \frac{dy}{dx} = 0 \)

ii) \( \frac{dy}{dx} > 0 \)

iii) \( \frac{dy}{dx} < 0 \).

14 The curve with equation \( y = 2x^2 + bx + 8 \) has a turning point at \( x = 3 \). Find the value of \( b \).

15 The curve with equation \( y = ax^2 + bx \) has a stationary point at \( (3, 9) \). Find the values of \( a \) and \( b \).

16 The curve with equation \( y = ax^2 + bx + c \) passes through the point \((2, -4)\) and has a stationary point at \((1, 2\frac{1}{2})\). Find the values of \( a, b \) and \( c \).

17 The curve with equation \( y = ax^3 - x^2 + bx + 2 \) has turning points at \( x = 1 \) and \( x = -2 \). Find the values of \( a \) and \( b \).

10C Solving maximum and minimum problems

There are many practical situations where it is necessary to determine the maximum or minimum value of a function. For quadratic functions, differentiation makes this a relatively simple task because, as we saw in the previous section, setting the derivative equal to zero
allows us to solve an equation to obtain the value(s) of $x$ for which the local maximum or minimum values (turning points) occur.

When solving maximum or minimum problems it should be verified that it is in fact a maximum or minimum by checking the sign of the derivative to left and right of the turning point.

In the case of cubic and higher order polynomials, the local maximum (or minimum) may or may not be the highest (or lowest) value of the function in a given domain.

An example where the local maximum, found by solving $f'(x) = 0$, is not the largest value of $f(x)$ in the domain $[a, b]$ is shown above. Here, $b$ is the point where $f(x)$ is greatest in this domain, and is called the absolute maximum for the interval.

**Case 1. The function is known**

**WORKED EXAMPLE 8**

A baseball fielder throws the ball so that the equation of its path is:

$$y = 1.5 + x - 0.02x^2$$

where $x$ (metres) is the horizontal distance travelled by the ball and $y$ (metres) is the vertical height reached.

**a** Find the value of $x$ for which the maximum height is reached (verify that it is a maximum).

**b** Find the maximum height reached.

**THINK**

1. Write the equation of the path.
2. Find the derivative $\frac{dy}{dx}$.
3. Solve the equation $\frac{dy}{dx} = 0$ to find the value of $x$ for which height is a maximum.
4. Determine the nature of this stationary point at $x = 25$ by evaluating $\frac{dy}{dx}$ to the left and right, say, at $x = 24$ and at $x = 26$.
5. Complete a gradient table and state the type of stationary point. The gradient changes from positive to negative as we move from left to right in the vicinity of $x = 25$.

**WRITE**

**a**

$$y = 1.5 + x - 0.02x^2$$

$$\frac{dy}{dx} = 1 - 0.04x$$

For stationary points: $\frac{dy}{dx} = 0$

$$1 - 0.04x = 0$$

$$-0.04x = -1$$

$$x = 25$$

When $x = 24$,

$$\frac{dy}{dx} = 1 - 0.04(24)$$

$$= 0.04$$

When $x = 26$,

$$\frac{dy}{dx} = 1 - 0.04(26)$$

$$= -0.04$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>24</th>
<th>25</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of $f''(x)$</td>
<td>+</td>
<td>0</td>
<td>–</td>
</tr>
<tr>
<td>Slope</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\therefore x = 25$ is a local maximum.

**b**

When $x = 25$,

$$y = 1.5 + 25 - 0.02(25)^2$$

$$= 14$$

So the maximum height reached is 14 m.
Case 2. The rule for the function is not given
If the rule is not given directly, then the steps below should be followed:
1. Draw a diagram if necessary and write an equation linking the given information.
2. Identify the quantity to be maximised or minimised.
3. Express this quantity as a function of one variable only (often this will be $x$).
4. Differentiate, set the derivative equal to zero and solve.
5. Determine, in the case of more than one value, which one represents the maximum or minimum value.
6. For some functions, a maximum or minimum may occur at the extreme points of the domain, so check these also.
7. Answer the question that is being asked.
8. Sketch a graph of the function if it helps to answer the question, noting any restrictions on the domain.

WORKED EXAMPLE 9
A farmer wishes to fence off a rectangular paddock on a straight stretch of river so that only three sides of fencing are required. Find the largest possible area of the paddock if 240 metres of fencing is available.

THINK
1. Draw a diagram to represent the situation, using labels to represent the variables for length and width, and write an equation involving the given information.
2. Write a rule for the area, $A$, of the paddock in terms of length, $l$, and width, $w$.
3. Express the length, $l$, of the rectangle in terms of the width, $w$, using equation [1].
4. Express the quantity, $A$, as a function of one variable, $w$, by substituting [3] into [2].
5. Solve $A'(w) = 0$.
6. Test to see if the stationary point at $w = 60$ will produce a maximum or minimum value for the area by evaluating $A'(w)$ to the left and right, say, at $w = 59$ and at $w = 61$.

WRITE/DRWA
Let $w = \text{width}$
Let $l = \text{length}$
Let $P = \text{perimeter}$

1. $P = l + 2w = 240$ [1]
2. $A = l \times w$ [2]
3. $l + 2w = 240$
   $l = 240 - 2w$ [3]

$A(w) = (240 - 2w)w$
$= 240w - 2w^2$

$A'(w) = 240 - 4w$

For stationary points: $A'(w) = 0$
$240 - 4w = 0$
$240 = 4w$
$w = 60$

When $w = 59$,
$A'(59) = 240 - 4(59)$
$= 4$

When $w = 61$,
$A'(61) = 240 - 4(61)$
$= -4$
Complete a gradient table and state the type of stationary point. The gradient changes from positive to negative as we move from left to right in the vicinity of \( w = 60 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>59</th>
<th>60</th>
<th>61</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f'(x) )</td>
<td>+</td>
<td>0</td>
<td>–</td>
</tr>
<tr>
<td>Slope</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \therefore \text{The stationary point is a local maximum.} \]

The area of the paddock is a maximum when \( w = 60 \).

\[ A(60) = (240 - 2 \times 60) \times 60 \]
\[ = 7200 \text{ m}^2 \]

Find the maximum area of the paddock by substituting \( w = 60 \) into the function for area.

**REMEMBER**

Defining a function and setting its derivative equal to zero to form an equation helps to tell us when a local maximum or minimum occurs. The solution(s) must then be substituted into the original function to find the actual maximum or minimum value(s).

**EXERCISE 10C**

**Solving maximum and minimum problems**

1. A golfer hits the ball so that the equation of its path is:
   \[ y = 1.2 + x - 0.025x^2 \]
   where \( x \) (metres) is the horizontal distance travelled by the ball and \( y \) (metres) is the vertical height reached.
   a. Find the value of \( x \) for which the maximum height is reached (and verify that it is a maximum).
   b. Find the maximum height reached.

2. If the volume of water, \( V \) litres, in a family’s hot water tank \( t \) minutes after the shower is turned on is given by the rule \( V = 200 - 1.2t^2 + 0.08t^3 \), where \( 0 \leq t \leq 15 \):
   a. find the time when the volume is minimum (that is, the length of time the shower is on)
   b. verify that it is a minimum by checking the sign of the derivative
   c. find the minimum volume
   d. use a CAS calculator to find the value of \( t \) when the tank is full again.

3. A ball is thrown into the air so that its height, \( h \) metres, above the ground at time, \( t \) seconds, after being thrown is given by the function:
   \[ h(t) = 1 + 15t - 5t^2 \]
   a. Find the greatest height reached by the ball and the value of \( t \) for which it occurs.
   b. Verify that it is a maximum.

4. A gardener wishes to fence off a rectangular vegetable patch against her back fence so that only three sides of new fencing are required. Find the largest possible area of the vegetable patch if she has 16 metres of fencing material available.
5 The sum of two numbers is 16.
   a By letting one number be \( x \), find an expression for the other number.
   b Find an expression for the product of the two numbers, \( P \).
   c Hence, find the numbers if \( P \) is a maximum.
   d Verify that it is a maximum.

6 The rectangle at right has a perimeter of 20 centimetres.
   a If the width is \( x \) cm, find an expression for the length.
   b Write an expression for the area, \( A \), in terms of \( x \) only.
   c Find the value of \( x \) required for maximum area.
   d Find the dimensions of the rectangle for maximum area.
   e Hence, find the maximum area.

7 A farmer wishes to create a rectangular pen to contain as much area as possible using 60 metres of fencing.
   a Write expressions for the dimensions (length and width) of the pen.
   b Hence, find the maximum area.

8 The cost of producing a particular toaster is \( $(250 + 1.2n^2) \) where \( n \) is the number produced each day. If the toasters are sold for $60 each:
   a write an expression for the profit, \( P \) dollars
   b find how many toasters should be produced each day for maximum profit
   c hence, find the maximum daily profit possible.

9 A company’s income each week is \( $(800 + 1000n - 20n^2) \) where \( n \) is the number of employees. The company spends $760 per employee for wages and materials.
   a Write an expression for the company weekly profit, \( P \).
   b Determine the number of employees required for maximum profit and hence calculate the maximum weekly profit.

10 The sum of two numbers is 10. Find the numbers if the sum of their squares is to be a minimum.

11 A square has four equal squares cut out of the corners as shown at right. It is then folded to form an open rectangular box.
   a What is the range of possible values for \( x \)?
   b In terms of \( x \) find expressions for the:
      i height
      ii length
      iii width of the box.
   c Write an expression for the volume, \( V \) (in terms of \( x \) only).
   d Find the maximum possible volume of the box.

12 The base and sides of a shirt box are to be made from a rectangular sheet of cardboard (measuring 50 cm \( \times \) 40 cm) with the corners cut out. Find:
   a the dimensions of the box required for maximum volume
   b the maximum volume.
   (Give answers correct to 2 decimal places.)

13 The volume of the square-based box shown at right is 256 cm\(^3\).
   a Find \( h \) in terms of \( l \).
   If the box has an open top find:
   b the surface area, \( A \), in terms of \( l \) only
   c the dimensions of the box if the surface area is to be a minimum
   d the minimum area. \((Hint: \frac{1}{l} = l^{-1}\).)

14 A closed, square-based box of volume 1000 cm\(^3\) is to be constructed using the minimum amount of sheet metal possible. Find its dimensions.
A cylindrical can, open at one end, is to be made out of aluminium. Use a CAS calculator if required to help answer the following.

a If the surface area of the can is to be 200 cm$^2$, find an expression for the height, $h$, of the can in terms of the radius, $r$.

b Find a function for the volume, $V$, of the can in terms of the radius, $r$.

c Find $\frac{dV}{dr}$.

d Hence, find the exact value of the radius that gives the maximum volume of the can. Give the radius correct to 2 decimal places.

e Find the volume of the can both in exact form and to the nearest cm$^3$.

A window is to be made with dimensions as shown in the diagram. It will consist of a semi-circle and a rectangle. Use a CAS calculator to help answer the following.

a If the perimeter of the window is 11 metres, show that $y = \frac{22 - 8x}{\pi + 2}$.

b Hence, find an expression for the area, $A$, of the window in terms of $x$.

c Find $\frac{dA}{dx}$.

d Hence, find the exact values of $x$ and $y$ such that the area of the window is a maximum.

e Find the exact maximum area of the window.

Applications of antidifferentiation

In the last chapter we found that, when the rate of change of a function (the derivative or gradient function) is known, we can determine the original function by a process called antidifferentiation.

If $f'(x) = x^n$, $n \in N$, then

$$f(x) = \frac{1}{n+1}x^{n+1} + c$$

where $c$ represents a constant.

This can be verified by differentiating

$$\frac{1}{n+1}x^{n+1} + c$$

The result is $x^n$.

Similarly, if $f'(x) = ax^n$, $a \in R$, $n \in N$, then

$$f(x) = \frac{a}{n+1}x^{n+1} + c$$

We saw previously that an alternative expression for the derivative was $\frac{dy}{dx}$. Likewise, if $\frac{dy}{dx} = x^n$, then

$$y = \frac{1}{n+1}x^{n+1} + c$$

The value of $c$ can be found if boundary conditions and given. Antidifferentiation can often be used to solve problems.

**WORKED EXAMPLE 10**

Find the rule for the function $f(x)$ if $f'(x) = 3 + 4x - x^2$ and $f(0) = 7$.

**THINK**

1 Write the given expression.

**WRITE**

$$f'(x) = 3 + 4x - x^2$$
Antidifferentiate \( f'(x) \) to obtain the general rule for \( f(x) \).

\[
f(x) = 3x + 2x^2 - \frac{x^3}{3} + c
\]

Substitute \( x = 0 \) and \( f(x) = 7 \) into \( f(x) \) and solve to find the value of the constant, \( c \).

\[
7 = 3(0) + 2(0)^2 - \frac{(0)^3}{3} + c
\]

\[
c = 7
\]

Write the rule for \( f(x) \).

\[
f(x) = 3x + 2x^2 - \frac{x^3}{3} + 7
\]

The pieces of information used to find the value of the constant that is generated following antidifferentiation are called **boundary conditions**.

**WORKED EXAMPLE 11**

The rate of change of the volume, \( V \) litres, of a balloon at any time, \( t \) seconds, after it is inflated beyond 6 litres is given by:

\[
\frac{dV}{dt} = 3t^2 - 8t + 1 \quad t \in [0, 3]
\]

**a** Express \( V \) as a function of \( t \).

**b** What is the volume of the balloon when \( t = 1 \)?

**THINK**

1. Write the given expression.

2. Find the general rule for volume by antidifferentiation.

3. Find the value of the constant, \( c \), by substituting \( t = 0 \) and \( V = 6 \) into \( V(t) \).

4. Write the rule for \( V(t) \).

**WRITE**

\[
\frac{dV}{dt} = 3t^2 - 8t + 1
\]

\[
V(t) = t^3 - 4t^2 + t + c
\]

\[
c = 6
\]

\[
V(t) = t^3 - 4t^2 + t + 6
\]

\[
V(1) = (1)^3 - 4(1)^2 + (1) + 6 = 4
\]

So the volume of the balloon at \( t = 1 \) is 4 litres.

An important application of antidifferentiation is in the study of kinematics (motion graphs). We will assume that all motion is along a straight line. The velocity can be antidifferentiated to find the displacement (distance or location from the origin) of the object. This process can be carried out by hand or using a CAS calculator (see Chapter 9, Worked example 24).

**WORKED EXAMPLE 12**

The velocity of a model train starting from 1 m to the right of the origin is given by \( v(t) = 6 - 2t \), where \( t \) is in seconds and \( v \) is in m/s for \( t \in [0, 5] \).

**a** When does it stop?

**b** What is its displacement at any time, \( t \)?

**c** How far is it from the starting point when \( t = 5 \)?

**d** Where is the train when it stops?

**e** How far does it travel in the first 5 seconds?
THINK  

a) The train stops when the velocity $v(t) = 0$. Solve for $t$ when $v(t) = 0$.

b) To find an equation for the displacement, find the antiderivative of the velocity.

c) We know that the model train begins 1 m to the right of the origin, which implies a displacement of 1 m or the point $(0, 1)$. To calculate $c$, substitute this point into the equation for displacement.

d) Write the equation for displacement.

e) The train starts 1 m to the right of the origin. It continues moving to the right until it stops 10 m to the right. It then moves back towards its starting position, but after 5 s, it is at 6 m to the right of the origin. Since distance has no direction, add the distances.

WRITE  

a) $v(t) = 0$
$6 - 2t = 0$
$\Rightarrow 2t = 6$
$\Rightarrow t = 3 \text{ s}$

b) Displacement, $x(t) = \int v(t) \, dt$
$x(t) = \int (6 - 2t) \, dt$
$x(t) = 6t - t^2 + c$
$(0, 1) \Rightarrow x(0) = 6(0) - (0)^2 + c = 1$
$c = 1$

$x(t) = 7t^2 + 6t + 1$

x(5) = (5)^2 + 6(5) + 1$
$x(5) = 6$
After 5 s the train is 6 m to the right of the origin.

x(3) = (3)^2 + 6(3) + 1$
$x(3) = 10$
The train stops 10 m to the right of the origin.

e) In the first 3 s the train moves from 1 m to the right to 10 m to the right, that is, 9 m. In the next 2 s, the train moves from 10 m to the right back to 6 m to the right, which is 4 m. The total distance is $9 + 4 = 13 \text{ m}$.

This example emphasizes the difference between displacement and the distance actually travelled.

The CAS calculator really becomes important when used to access functions more difficult to antidifferentiate.

In the next example the CAS calculator is used for the antidifferentiation. But the rest of the working has been completed without technology because in this case it is quicker and easier to do so. The CAS calculator method (see Chapter 9, Worked example 24) is included for comparison. It will be important in formal assessment, when time is restricted, to decide the quickest and best approach. CAS should be used in situations when it is the better (or the only) option.

WORKED EXAMPLE 13

The velocity of a toy car is given by $v(t) = \frac{6}{(t + 3)^2} + 1$, where $t$ is in seconds and $v(t)$ is in m/s for $t \in [0, 4]$. If the car starts at $x = 0$, use a CAS calculator to help you answer the following.

a) What is its displacement at any time $t$?

b) How fast was the car moving at the start?

c) How far is the train from the starting point when $t = 2$?

d) Show that the car did not stop.
To find an equation for the displacement, find the antiderivative of the velocity.

On the Main screen, complete the entry line as:
\[
\int \left( \frac{6}{(t + 3)^2} + 1 \right) dt
\]
Then press \( \boxed{} \).

2. We know that the toy car begins its journey at the origin, that is, at the point \((0, 0)\). To calculate \( c \), substitute this point into the equation for displacement.

3. Write the equation for displacement.

\( \text{b} \) The distance can be calculated by substituting \( t = 2 \) into the equation for displacement.

\( \text{c} \) If the car did not stop, then its velocity \( \neq 0 \).
Show that \( v(t) \neq 0 \) for \( t \in [0, 4] \) by sketching the graph with the domain \([0, 4]\).

On the Graph & Tab screen, enter:
\[
y_1 = \frac{6}{(x + 3)^2} + 1 \mid 0 \leq x \leq 4
\]
Tick \( y_1 \) and tap \( \text{[Enter]} \).

\( v(t) \neq 0 \) for \( t \in [0, 4] \)

Remember

When finding the antiderivative of a function, for each term in the function increase the power of the variable by one and divide by the resulting power. Add a constant.
1. WEIO Find the rule for the function \( f(x) \) if \( f'(x) = 3x^2 - 2x \) and \( f(2) = 0 \).

2. If \( f'(x) = 3 + 5x - 2x^2 \) and the y-intercept is 7, find \( f(x) \).

3. The y-intercept of a curve is 10 and \( \frac{dy}{dx} = (x + 1)(x - 3) \). Find the value of \( y \) when \( x = 3 \).

4. MC If the gradient function of a curve that passes through the point (2, 2) is \( f'(x) = 2x - 5 \), then the function \( f(x) \) is:
   A. \( x^2 - 5x + 8 \)  
   B. \( x^2 - 5x - 1 \)  
   C. \( x^2 - 5 \)  
   D. \( x^2 - 5x \)  
   E. \( x^2 - 2 \)

5. MC If \( f'(x) = 4x + 1 \) and the y-intercept is -3 then \( f(x) \) equals:
   A. \( x^2 + 2x - 3 \)  
   B. \( 2x^2 + x - 1 \)  
   C. \( 2x^2 + x - 3 \)  
   D. \( 2x^2 + 2x - 1 \)  
   E. \( x^2 + x \)

6. MC A curve passes through the point (2, 1) and has a gradient function \( f'(x) = x(3x - 5) \). The function must be:
   A. \( f(x) = x^3 - 3x^2 + 5 \)  
   B. \( f(x) = x^3 - \frac{5}{2}x^2 + 2 \)  
   C. \( f(x) = 3x^2 - 5x - 1 \)  
   D. \( f(x) = x^3 - \frac{5}{2}x^2 + 3 \)  
   E. \( f(x) = \frac{3}{4}x^4 - \frac{5}{2}x^3 + 9 \)

7. WEII The velocity (\( v \)) of an aircraft is changing as it accelerates. Its acceleration (rate of change of velocity) at any time, \( t \), after it begins accelerating from rest along a runway is given by \( \frac{dv}{dt} = 6t^2 - 4t + 5 \), where \( v \) is in km/h and \( t \) is in seconds.
   a. Express \( v \) as a function of \( t \).
   b. Find the velocity after 5 seconds.

8. The rate of change of position (velocity) of a particle travelling in a straight line is given by \( \frac{dx}{dt} = t^2 - 6t + 2 \), where \( x \) is in metres and \( t \) is in seconds. If the particle starts at \( x = 1 \), find its position when \( t = 3 \).
9 The rate of increase of volume per unit increase in depth for a particular container is given by:

\[
\frac{dV}{dh} = 2(h + 5)^2
\]

where \( V \) cm\(^3\) is the volume and the depth is \( h \) cm.

a. If \( V = 0 \) when \( h = 0 \), express \( V \) as a function of \( h \).

b. Find the volume at a height of 7 cm.

10 The weekly rate of change of profit with respect to the number of employees, \( n \), in a factory is:

\[
\frac{dP}{dn} = 3.182 - \frac{3}{4} \sqrt{n}
\]

where \( P \) is in thousands of dollars.

a. Find the number of employees for maximum profit (assume \( P = 0 \) when \( n = 0 \)).

b. Hence find the maximum profit.

11 The rate of deflection from the horizontal of a 2 m long diving board when a 70 kg person is \( x \) m from its fixed end is:

\[
\frac{dy}{dx} = -0.06(x + 1)^2 + 0.06
\]

a. What is the deflection, \( y \), when \( x = 0 \)?

b. Find the equation that measures the deflection at any point on the board.

c. Find the maximum deflection. (Be careful.)

12 The rate of change of height of a hot-air balloon is given by

\[
\frac{dh}{dt} = 4t - 1
\]

where \( h \) is the height above the ground in metres after \( t \) seconds.

a. Write \( h \) as a function of \( t \).

b. Find the height after 4 s.

c. How long does it take the balloon to reach a height of 60 m?

d. Where is the object when it stops?

e. How far does it travel in the first 4 s?

13 WE12 The velocity of an object starting from 3 m to right of the origin is given by

\[ v(t) = 12 - 6t, \]

where \( t \) is in seconds and \( v(t) \) is in metres per second for \( t \in [0, 6] \)

a. When does it stop?

b. What is its displacement at any time \( t \)?

c. How far is it from the starting point when \( t = 4 \)?

d. Where is the object when it stops?

e. How far does it travel in the first 4 s?

14 WE13 The velocity of a cyclist starting from 2 km to left of the origin is given by

\[ v(t) = 10 - 4t, \]

where \( t \) is in hours and \( v(t) \) is in kilometres per hour for \( t \in [0, 3] \).

a. What is its displacement at any time \( t \)?

b. How far is it from the starting point when \( t = 3 \)?

c. When and where is the train when it stops?

d. How far does it travel in the first 3 s?
15 The velocity upwards of a hot-air balloon starting 2 m above the ground is given by
\( v(t) = 4t - t^2 \), where \( t \) is in seconds and \( v(t) \) is in metres per second for \( t \in [0, 5] \).
   a. When was it stationary?
   b. What was its displacement at any time \( t \)?
   c. How far was it from the starting point when \( t = 5 \)?
   d. Where was the balloon when it stopped?
   e. How far did it travel in the first 5 s?

16 The acceleration, \( \frac{dv}{dt} \), of a skier starting from 6 m to right of the origin with a velocity of 7.5 metres per second is given by \( a(t) = 6 - 3t \), where \( t \) is in seconds and \( a \) is in metres per second squared for \( t \in [0, 10] \).
   a. What is its velocity at any time \( t \)?
   b. What is its displacement at any time \( t \)?
   c. How far is it from the starting point when \( t = 7 \)?
   d. When and where is the skier stationary?
   e. How far did the skier travel in the first 7 seconds?

17 The velocity of a jogger is given by \( v(t) = \frac{4}{(t+2)^2} \), where \( t \) is in seconds and \( v(t) \) is in metres per second for \( t \in [0, 5] \). After 2 s the jogger is 4 m to the right of the origin. Use a CAS calculator to help you answer the following.
   a. What was the jogger’s displacement at any time \( t \)?
   b. Where did the jogger start from?
   c. How fast was the jogger running at the start?
   d. How far was the jogger from the starting point when \( t = 3 \)?
   e. Show that the jogger didn’t stop.
   f. How far did the jogger travel in the first 3 s?

18 The velocity of a toy remote-controlled racing car is given by \( v(t) = \frac{6}{(t+1)^2} - 6 \), where \( t \) is in seconds and \( v(t) \) is in metres per second for \( t \in [0, 4] \). At the start it was 5 m to the left of the origin. Use a CAS calculator to help you answer the following.
   a. What was its displacement at any time \( t \)?
   b. How fast was the car moving at the start?
   c. How far was the car from the starting point when \( t = 4 \)?
   d. Show that the car did not stop.
   e. How far did the car travel in the first 4 seconds?

19 The velocity of a walker is given by \( v(t) = 10 \cos \left( \frac{\pi(t-2)}{4} \right) \), where \( t \) is in seconds and \( v(t) \) is in metres per second for \( t \in [0, 5] \). At \( t = 0, x = -2 \) m. Use a CAS calculator to help you answer the following.
   a. What was its displacement at any time \( t \)?
   b. How fast was the walker moving at the start?
   c. How far (correct to 2 decimal places) was the walker from the starting point when \( t = 5 \)?
   d. Where (correct to 2 decimal places) and when did the walker stop?
   e. How far did the walker travel in the first 5 s?
Rates of change

- Average rate of change is \( \frac{\text{change in } y}{\text{change in } x} \).

- The derivative of a function, \( f'(x) \), or \( \frac{dy}{dx} \), is needed in order to calculate the (instantaneous) rate of change at a particular point. The rate of change of a function, \( f(x) \), at \( x = a \) is given by \( f'(a) \).

Sketching graphs containing stationary points

- Stationary points occur when \( f'(x) = 0 \).
- Three types of stationary point exist, and by testing the sign of the gradient to the left and right of a stationary point, the nature (type) of the stationary point can be determined:
  1. local maximum turning points (\( f'(x) \) changes from + to – moving left to right)
  2. local minimum turning points (\( f'(x) \) changes from – to + moving left to right)
  3. stationary points of inflection (the sign of \( f'(x) \) remains the same on both sides moving left to right).

Solving maximum and minimum problems

- By solving the equation \( f'(x) = 0 \) and substituting the solutions into the original function, the maximum or minimum value of a quantity may be found. When the function is not provided, it is necessary to formulate a rule in terms of one variable using the information given. Drawing a diagram to represent the situation is often useful.
- Always test to determine if a stationary point is a maximum or a minimum by checking the sign of the gradient to the left and right of the point.
- Check whether or not the local maximum or minimum is the absolute maximum or minimum. The absolute maximum or minimum may be the value of the function at one end of a specified interval.

Applications of antidifferentiation

- When the derivative of a function is known, antidifferentiation can provide the original function. Since the original function may have contained a ‘constant’, this must be allowed for, and can be found using the boundary conditions provided in the question.
- Antidifferentiation can be applied to kinematics (motion graphs), especially when calculating distances travelled.
1. If the position of a particle moving in a straight line is given by the rule \( x(t) = -2t^2 + 8t + 3 \), where \( x \) is in centimetres and \( t \) is in seconds, find:
   a. the initial position of the particle
   b. the rate of change of displacement (that is, the velocity) at any time, \( t \)
   c. the rate of change when \( t = 4 \)
   d. when and where the velocity is zero
   e. whether the particle is moving to the left or to the right when \( t = 3 \)
   f. the distance travelled in the first 3 seconds.

2. For the function \( f(x) = x^3 - 3x + 2 \):
   a. find the \( y \)-intercept
   b. find the \( x \)-intercepts
   c. find the stationary points and state their type
   d. sketch the graph of \( f(x) \).

3. The curve with equation \( y = ax^2 + bx + c \) passes through the point \((0, -35)\) and has a zero gradient at the point \((5, 40)\). Find the values of \( a \), \( b \) and \( c \).

4. If the volume of liquid in a vat, \( V \) litres, during a manufacturing process is given by \( V = 6t - t^2 \), where \( t \in [0, 6] \), find:
   a. the rate of change 2 hours after the vat starts to fill up
   b. when the vat has a maximum volume.

5. If a piece of wire is 80 cm long:
   a. find the area of the largest rectangle that can be formed by the wire
   b. determine whether a circle would give a larger area.

6. Find the maximum possible volume of a fully enclosed cylindrical water tank given that the total internal surface area of the tank is 600 \( \pi \) square units.

7. The rate of increase of height, \( h \) metres, of an ascending helicopter at any time, \( t \) minutes, after it takes off is \( \frac{dh}{dt} = t^2 - 14t + 45 \).
   a. Find an expression for the height at any time.
   b. Find the height 6 minutes after takeoff.
   c. Find the maximum height reached in the first 9 minutes.

8. A particle travels such that its velocity at any time, \( t \), is given by \( v = 2t + 1 \).
   a. Given that velocity represents the rate of change of position, \( x \), write down the relationship between \( v \) and \( x \).
   b. If \( x = 3 \) when \( t = 2 \), write an expression for \( x \) in terms of \( t \).
   c. Find the position of the particle when \( t = 10 \).

9. A robot moves in a straight line, starting 14 m to the right of the origin. Its velocity at any time, \( t \in [0, 5] \), is given by \( v(t) = 6t^2 - 2t \), where \( v(t) \) is in metres per second and \( t \) is in seconds.
   a. What was the starting velocity of the robot?
   b. What was the displacement at any time \( t \)?
   c. When and where was the robot stationary?
   d. What was its location at \( t = 3 \)?
   e. How far did it travel in the first 3 seconds?

**Multiple Choice**

1. The rate of change of \( f(x) = 2x^3 - 5x^2 + 7 \) when \( x = 2 \) is:
   A. \(-4\) B. 7 C. \(-36\) D. 0 E. 4

2. If \( V = -3r^2 + 7t + 50 \) then the average rate of change between \( t = 1 \) and \( t = 4 \) is:
   A. \(-10\) B. \(-10\) C. \(-6\) D. \(-8\) E. 0

3. If \( f(x) = 5 + 15x + 6x^2 - x^3 \), then the gradient is zero when \( x \) equals:
   A. 1 or \(-5\) B. 1 or 5 C. \(-1\) or 5 D. \(-1\) or \(-5\) E. 0 or \(-1\)

4. The curve \( y = x^2 - 10x + 21 \) has:
   A. a local maximum at \((5, 0)\)
   B. a local minimum at \((5, -4)\)
   C. a stationary point of inflection at \((5, 0)\)
   D. a local maximum at \((5, -4)\)
   E. a stationary point of inflection at \((5, -4)\)
5 When \( x = -2 \), the graph of \( y = 2x^2 + 3x - 5 \):
- A is increasing
- B has a local maximum
- C has a stationary point of inflection
- D has a local minimum
- E is decreasing

6 For a particular function \( g(x) \), \( g(1) = 0 \) and \( g'(x) < 0 \)
if \( x \geq 1 \). The graph that could represent \( g(x) \) is:

   ![Graphs of functions](image)

7 The maximum value of \( f(x) = -2x^2 + 8x \) is:
- A 40
- B 0
- C 4
- D -24
- E 8

8 The local minimum value of \( h(x) = \frac{1}{3}x^3 + 6x^2 - 28x - 3 \) occurs when \( x \) equals:
- A 2
- B -4
- C 0
- D -3
- E 1

9 For the function \( y = x^3 - 6x^2 + 9x - 4 \), the values of \( x \) for which \( \frac{dy}{dx} < 0 \) are:
- A \( x < 1 \)
- B \( 1 < x < 4 \)
- C \( 1 < x < 3 \)
- D \( x < 1 \cup x > 3 \)
- E \( x > 3 \)

10 The function \( g(x) = (x + 3)^3 \) has:
- A a local maximum when \( x = -3 \)
- B a stationary point of inflection when \( x = -3 \)
- C a local minimum where \( x = -3 \)
- D a local minimum where \( x = 3 \)
- E a stationary point of inflection where \( x = 3 \)

11 A curve with a local maximum and a local minimum is:
- A \( y = x^3 + 2x^2 - 7x + 1 \)
- B \( y = x^2 - 3x + 1 \)
- C \( y = x^3 + 7 \)
- D \( y = (x - 2)^3 \)
- E \( y = x^2 + 6x \)

12 The antiderivative of \( 12x + 3 \) is:
- A \( 6x^2 + 3x \)
- B \( 24x^2 + 3x + c \)
- C \( 24x^2 + 3x \)
- D \( 6x^3 + 3x + c \)
- E \( 6x^2 \)

13 If the gradient of a curve is \( \frac{dy}{dx} = (x - 2)(x + 5) \) and its y-intercept is \(-3\), then its rule is:
- A \( y = \frac{1}{3}x^3 + \frac{2}{3}x^2 - 10x - 3 \)
- B \( y = \frac{1}{3}x^3 + \frac{2}{3}x^2 - 5x - 3 \)
- C \( y = x^3 + 3x^2 - 10x - 3 \)
- D \( y = \frac{1}{3}x^3 + \frac{2}{3}x^2 - 10x - 10 \)
- E \( y = \frac{1}{4}x^4 - 10x^2 \)

**EXTENDED RESPONSE**

1 A ball is thrown vertically up so that its height above the ground, \( h \) metres, at any time, \( t \) seconds, after leaving the thrower’s hand is given by the function \( h(t) = \frac{8}{3}t^3 - \frac{8}{9}t^2 + 2 \).

   a Find the height of the ball as it leaves the thrower’s hand.
   b Find when and where the ball reaches its greatest height.
   c Find when the ball returns to the same level that it left the thrower’s hand.
   d If the ball isn’t hit, find when the ball hits the ground to the nearest thousandth of a second.
   e Hence, state the domain and range of \( h(t) \).
   f Sketch the graph of \( h \) versus \( t \).

2 A piece of wire 100 cm long is to be cut so that one piece is used to form a square, while the other is used to form a circle. If the edge length of the square is \( x \) cm, find, in terms of \( x \),
Chapter 10 Applications of Differentiation

3 An observer on the ground initially sights an aircraft at an altitude of approximately 2 km, diving towards the Earth. The aircraft’s altitude in metres is given by the equation

\[ f(t) = -\frac{11}{8}t^3 + 50t^2 - 560t + 2200, \]

where \( t \) is the time in seconds after the aircraft is first sighted. (Give answers to 1 decimal place.)

a Find the actual altitude of the aircraft when it is first sighted.

b Calculate the average rate of change of the aircraft’s altitude over the first 3 seconds.

c Write an expression for the derivative \( f'(t) \).

d Calculate the instantaneous rate of change of the aircraft’s altitude after 3 seconds.

e Based on your answers to parts b and d, is the aircraft pulling out of the dive, or is its situation worsening?

f After several seconds the pilot manages to stabilise the aircraft and its altitude begins to increase. At what time does this occur, and how far is the aircraft above the ground?

g Despite stabilising the aircraft, the pilot decides to eject, but the minimum altitude at which this can be safely attempted is 400 m above the ground. What is the maximum altitude achieved before the aircraft goes back into a downward path?

h The pilot actually ejects at \( t = 16 \) s. Explain whether or not the aircraft has sufficient altitude to make this safe.

i Using a CAS calculator, find how soon after ejection the aircraft will crash.

4 The Pantheon is an ancient building located in Rome, Italy. The main structure consists of a hemispherical dome that has an 8.7-m oculus (hole) at the top and is supported by a 6-m-thick cylindrical wall. It is known that the volume of the structure is the maximum possible given its internal surface area.

a Ignoring the oculus, write an expression for the surface area of the structure (including a circular base) in terms of the radius of the dome, \( r \), and the height of the cylindrical wall, \( h \).

b If the internal surface area of the structure is 7362 m\(^2\), express \( h \) in terms of \( r \).

c Write an expression for the volume of the structure in terms of \( r \).

d Show that the height of the Pantheon’s cylindrical wall is the same as the diameter of its base.

5 The function \( y = \frac{a}{3}x^3 - \frac{b}{2}x^2 + 6x + c \) has turning points at \( x = 1 \) and \( x = -1 \).

a Write an expression for the derivative \( \frac{dy}{dx} \).

b Determine the values of \( a \) and \( b \).

c Find the equation of the derivative \( \frac{dy}{dx} \).

d Determine the nature of the turning point at \( x = 1 \).

e Determine the nature of the turning point at \( x = -1 \).

f If the original function touches the \( x \)-axis at \( x = 1 \), find the value of the constant, \( c \), and hence determine the equation of the original function.

6 The function given by \( f(x) = a^3x^2 - a^2x^3 \), where \( a \neq 0 \), has a turning point at the point \( T(b, c) \), where \( b \neq 0 \).

a Find \( f'(x) \).

b Show that \( b = \frac{3}{2}a \).

c Express \( c \) in terms of \( a \), and hence state the coordinates of \( T \).

d If \( a = \frac{3}{2} \), find the coordinates of the turning point and explain why it is a local maximum.

7 The local council has decided to connect two parallel bicycle paths 30 m apart with a curved bitumen path between A and B. A keen amateur mathematician decides that the path should consist of two connected parabolas, (with turning points at \( (0, 0) \) and \( (40, 30) \) respectively), with a smooth connection (same gradient) at the point of intersection. The axes are placed as shown in the diagram. The lower parabola needs to cross a bridge over a creek at \( (20, 10) \) and so this part of the path cannot change.
a Find the equation for the lower parabola.

b Show that the upper parabola has the equation \( y = a(x - 40)^2 + 30 \).

c Find the equation of the upper parabola, assuming the two parabolas meet at the bridge (20, 10).

d Show that the connection is not smooth.

e For a smooth connection the two parabolas must meet elsewhere. Show that, for the paths to meet at

\[
(x, y), \quad \frac{x^2}{40} = a(x - 40)^2 + 30\]

where \( a \) is as defined in part b; hence, \( a = \frac{x^2 - 1200}{40(x - 40)^2} \).

f If the connection is to be smooth, show that \( a = \frac{x}{40(x - 40)} \).

g Solve the two simultaneous equations with or without technology and show that the paths meet at

(30, 22.5) when \( a = -0.075 \).

8 A bungee jumper leaped from a platform 30 m above a cold deep stream. If the jumper extended her arms, her fingertips reached 2.1 m above her feet. Taking the platform as the origin (where her feet are attached to the elastic cord) and upwards as positive, her velocity was given by \( v(t) = 5(e^{-t^2} - e^{-2t}) \), where \( t \) is in seconds and \( v \) is in metres per second. Answers should be expressed to 2 decimal places if an exact whole number is not obtained.

a Where were her feet when \( t = 0 \)?

b What was the displacement at any time \( t \)?

c When and where was the velocity zero?

d Where were her feet after 3 seconds and how fast was she moving?

e How far did she travel in 3 seconds?

f If she fell head first vertically downwards with arms outstretched, did she get wet?

g If so, how much of her was wet and, if not, how much short of the water was she?
Chapter 10 Applications of Differentiation

Chapter opener

Digital doc
• 10 Quick Questions: Warm-up with ten quick questions on applications of differentiation (page 449)

10A Rates of change

Tutorial
• WE2 int-0321: Find the rate of change of the height of a javelin at specific times and when the height reaches 17 metres (page 451)

Digital docs
• SkillSHEET 10.1: Practise calculating average rates of change (page 453)
• Spreadsheet 048: Investigate gradients between two points on a graph using a spreadsheet (page 453)
• SkillSHEET 10.2: Practise calculating instantaneous rates of change (page 453)

10B Sketching graphs containing stationary points

Tutorial
• WE6 int-0991: Find the stationary points and determine their nature, the coordinates of all axial intercepts and sketch the graph of a cubic (page 458)

Digital docs
• Spreadsheet 108: Investigate quadratic graphs using a spreadsheet (page 460)
• Spreadsheet 014: Investigate cubic graphs using a spreadsheet (page 460)
• SkillSHEET 10.3: Review of discriminant (page 462)
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10C Solving maximum and minimum problems

Tutorial
• WE9 int-0323: Find the largest possible area of a paddock given 240 metres of fencing (page 464)

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• Spreadsheet 108: Investigate quadratic graphs using a spreadsheet (page 465)
• Spreadsheet 014: Investigate cubic graphs using a spreadsheet (page 465)

10D Applications of Antidifferentiation

Interactivity
• Area under curves int-0269: Consolidate your understanding of area under curves (page 467)

Tutorial
• WE12 int-0324: Determine the displacement at any time t of a train, where it stops and the distance it travels in the first five seconds (page 468)

Digital doc
• WorkSHEET 10.2: Apply differentiation skills to a variety of problems (page 472)

Chapter review

Digital doc
• Test Yourself: Take the end-of-chapter test to test your progress (page 478)

To access eBookPLUS activities, log on to www.jacplus.com.au
20 minutes

1 Function \( f \) is defined as \( f : R \rightarrow R \), where \( f(x) = 2x^2 + 3x \).

   a Determine \( f(2 + h) \). 
   b Hence, determine the gradient of the tangent at \( x = 2 \) using first principles.

2 The distance, \( d \), in metres, over time, \( t \), in seconds, a battery-powered toy car travelled is shown in the graph below.

   a Describe the car’s speed during the first 30 s.
   b Determine the exact average speed, in m/s, between \( t = 30 \) s and \( t = 75 \) s.
   c Using your answer from part b, determine car’s instantaneous speed, in m/s at \( t = 50 \) s.

3 The function \( g(x) \) is shown below.

\[ g(x) = \begin{cases} 
-2, & x < -1 \\
4, & x \geq -1 
\end{cases} \]

On the same set of axes, sketch \( g'(x) \).

4 A function, \( f \), passes through the point \( (1, 5) \) and has a gradient function \( f'(x) = 4 \). Determine the function \( f(x) \).

5 The equation of the tangent to the curve \( f(x) = x^3 - 2x^2 - 3x + 2 \) at the point where the curve crosses the \( y \) axis is:

   a \( y = -4.28x + 2.26 \)
   b \( y = -3.52x + 2.04 \)
   c \( y = 7.78x + 10.45 \)
   d \( y = 9.49x - 26.71 \)
   e \( y = 2x^2 - 4x - 3 \)

6 A water tank is being filled with water at a constant rate of \( \frac{dy}{dt} \) litres/minute. Water is removed from the tank at the rate 10 litres/minute. Which one of the following will determine the volume of water in litres, \( V \), in the tank at any time, \( t \), in minutes?

   a \( V = \int V \, dt - 10 \)
   b \( V = \int \left( \frac{dV}{dt} - 10 \right) \, dt \)
   c \( V = \int \left( \frac{dV}{dt} \right) + 10 \)
   d \( V = \int \left( \frac{dV}{dt} \right) - 10 \)
   e \( V = \int \left( 10 - \frac{dV}{dt} \right) \, dt \)
1 A cross-section of the Black Range is shown in the diagram below.

![Diagram of a cross-section of the Black Range showing the ridge and valley with specific coordinates.]

The shape of the ridge can be described by the cubic equation \( h(x) = ax^3 + bx^2 + cx + d \), where \( x \) is the horizontal distance in metres and \( h(x) \) is the height in metres about sea level.

a Show that \( d = 150 \).  

b Determine the equation for \( h'(x) \) in terms of \( a, b \) and \( c \). 

c Two stationary points exist at \( x = 40.73 \) and \( 382.60 \). Using your answer from part b, write down two equations in terms of \( a, b \) and \( c \). 

d The point (550, 150) lies on the ridge. Write down an equation that would enable a third simultaneous equation to be used to find the values of \( a, b \) and \( c \). 

e The value of \( a \) is \(-0.00001\). Using any two equations found in previous parts, determine the values of \( b \) and \( c \) correct to 4 decimal places. 

2 A circular enclosure of radius \( r \), in metres, and a square enclosure are made from a 300 m length of fencing wire. To form the circular enclosure a length of \( x \) m is cut from the 300 m length. All of the wire is used to form the two enclosures.

a Show that \( r = \frac{x}{2\pi} \). 

b The remaining wire is made into a square of side length \( \frac{300-x}{4} \) m. Determine the area of the square enclosure in terms of \( x \). 

c Write down the equation that determines the total area \( A \) of the two enclosures. 

d In the context of this problem, write down the feasible domain. 

e Write down the equation for \( \frac{dA}{dx} \). 

f Determine the exact value of \( x \) so that the area of both enclosures will be a minimum. 

g Using your answer from part e, show that area is a minimum.