Euclidean geometry (extension)
This chapter is included as part of the geometry requirement of the additional area of study in the *extensions of Reasoning and Data.* It includes some of the material considered in Chapter 11 of *Space and Number.* However, the treatment in this chapter is more formal than that in *Space and Number,* and formal proofs of angle theorems, circle theorems, concurrency theorems, and Pythagoras' theorem and its converse, are provided.

Throughout this chapter, we will make certain statements without proving them. Such statements are called *axioms.* Statements that are proved to be true are called *theorems.*

An *axiom* is a statement that is accepted as true without proof.

A *theorem* is a statement that is proved to be true.

If the straight lines $AB$ and $CD$ intersect at $O$, four angles are formed. Any two of these that are not adjacent are *vertically opposite.*

$\angle AOD$ and $\angle BOC$ are vertically opposite.

$\angle AOC$ and $\angle BOD$ are vertically opposite.

$\angle AOC$ and $\angle AOD$ are adjacent (and also supplementary).

$\angle AOC$ and $\angle COB$ are adjacent (and also supplementary).

Two angles are supplementary if their sum is $180^\circ$. For example, $\angle AOC$ and $\angle AOD$ are supplementary in Figure 15-1.

### 15.1 Assumptions

We make the following assumptions concerning angles:

1. If two straight lines intersect:
   
   a. the sum of the adjacent angles is $180^\circ$.
      
      \[ a^\circ + b^\circ = 180^\circ \quad a^\circ + c^\circ = 180^\circ \]  \hspace{1cm} (Figure 15-1)

   b. the vertically opposite angles are equal.
      
      \[ b^\circ = c^\circ \quad \angle AOD = \angle COD \]  \hspace{1cm} (Figure 15-1)

### Parallel lines

Two lines, $AB$ and $CD$, are *parallel* if they are in the same plane and do not intersect. We write $AB \parallel CD$ to indicate that the lines $AB$ and $CD$ are parallel.

The line $PQ$, which cuts across the parallel lines, is called a *transversal.*

In Figure 15-2:

$\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, and $\angle 4$ and $\angle 8$ are pairs of *corresponding angles.*

$\angle 2$ and $\angle 8$, and $\angle 3$ and $\angle 5$ are pairs of *alternate angles.*

$\angle 2$ and $\angle 5$, and $\angle 3$ and $\angle 8$ are pairs of *co-interior angles.*
2. If two lines are parallel and they are intersected by a transversal, then:
   a. corresponding angles are equal
   b. alternate angles are equal
   c. co-interior angles are supplementary.

3. We will also assume the converse of Assumption 2. If two lines are intersected by a transversal, then the lines are parallel if:
   a. corresponding angles are equal
   b. alternate angles are equal, or
   c. co-interior angles are supplementary.

15.2 Angle properties of a triangle

**Theorem 1**

The sum of the angles of a triangle is 180°.

![Diagram](image)

Given: \( ABC \) is any triangle.

To prove: \( a + b + c = 180 \).

Construction:

Through \( B \), draw \( XY \) parallel to \( AC \).

**Proof:**

\[
\begin{align*}
  x &= a \quad \text{(Alternate angles equal)} \\
  y &= c \quad \text{(Alternate angles equal)} \\
  x + b + y &= 180 \quad \text{(Straight angle is 180°)} \\
\therefore \quad a + b + c &= 180
\end{align*}
\]

**Theorem 2**

An exterior angle of a triangle is equal to the sum of the two remote interior angles.

![Diagram](image)

Given: \( \triangle ABC \) with \( AC \) produced through \( D \).

To prove: \( e = a + b \).

**Proof:**

\[
\begin{align*}
  e + c &= 180 \quad \text{(Straight angle 180°)} \\
  a + b + c &= 180 \quad \text{(Angle sum of } \triangle \text{ is 180°)} \\
\therefore \quad e &= a + b
\end{align*}
\]
An *isosceles* triangle is one with two sidelengths equal. It can be proved that the base angles are equal, i.e. \( \angle B = \angle C \), and that the line drawn from the vertex, \( A \), to the midpoint, \( X \), of the base is an axis of symmetry. As a consequence:
\[
BX = XC \quad \text{and} \quad \angle AXB = \angle AXC = 90^\circ.
\]

![Figure 15-5](image1)

(These properties will be proved later, in Example 3).

The converse can also be proved to be true, i.e. if two angles of a triangle are equal, then the sides opposite the angles are equal in length.

An *equilateral* triangle is a particular case of the isosceles triangle in which the *three* sidelengths are equal. It can be proved that the magnitude of each angle is 60°.

![Figure 15-6](image2)

**Example 1**

In Figure 15-7, if \( DA = DB = DC \), prove that \( \angle ABC \) is a right angle.

![Figure 15-7](image3)

**Proof:**

\[
\begin{align*}
\angle DAB &= \angle DBA \quad (\triangle DAB \text{ is isosceles}) \\
\angle DCB &= \angle DBC \quad (\triangle DCB \text{ is isosceles})
\end{align*}
\]

In \( \triangle ABC \):
\[
2a + 2b = 180 \quad \text{(angle sum of } \triangle = 180^\circ) \\
\therefore \quad a + b = 90 \\
\therefore \quad \angle ABC = 90^\circ
\]
Example 2

In the diagram below, given that $BA \parallel DE$, $AB = BC$, $CD = DE$, and $B$, $C$ and $D$ are collinear, prove that $\angle ACE = 90^\circ$.

![Figure 15-8](image)

The word 'collinear' means 'in the same straight line'.

**Proof:**

In $\triangle ABC$:

$\angle BCA = \angle BAC$ (isosceles)

$\angle ABC + \angle BCA + \angle BAC = 180^\circ$ (angle sum of $\triangle$)

i.e. $\angle ABC + 2\angle BCA = 180^\circ$ .................................................. (1)

In $\triangle CDE$:

$\angle ECD = \angle CED$ (isosceles)

$\therefore \angle CDE + \angle ECD + \angle CED = 180^\circ$ (angle sum of $\triangle$)

i.e. $\angle CDE + 2\angle ECD = 180^\circ$ .................................................. (2)

Add (1) and (2):

$\therefore \angle ABC + 2\angle BCA + \angle CDE + 2\angle ECD = 360^\circ$

But:

$\angle ABC + \angle CDE = 180^\circ$ (co-interior angles)

$\therefore 2\angle BCA + 2\angle ECD = 180^\circ$

$\therefore \angle BCA + \angle ECD = 90^\circ$

$\therefore \angle ACE = 90^\circ$ (straight angle at $C$ is $180^\circ$)

**Alternative approach**

![Figure 15-9](image)

Construction:

Through $C$, draw $CF \parallel BA$ and $DE$.

**Proof:**

In $\triangle ABC$:

$\angle BCA = \angle BAC = a^\circ$ (isosceles)

$\angle FCA = \angle BAC = a^\circ$ ($BA \parallel CF$)

In $\triangle CDE$:

$\angle ECD = \angle CED = b^\circ$ (isosceles)

$\angle FCE = \angle CED = b^\circ$ ($DE \parallel CF$)

$\therefore$ Around the point $C$, we have:

$2a^\circ + 2b^\circ = 180^\circ$ (straight angle at $C$ is $180^\circ$)

$\therefore a^\circ + b^\circ = 90^\circ$

$\therefore \angle ACE = 90^\circ$. 
Exercises 15a

1 Two straight lines, $PQ$ and $XY$, intersect at a point, $R$. If the rays $RM$ and $RN$ bisect $\angle QRY$ and $\angle PRY$ respectively, prove that $\angle NRM$ is a right angle.

2 From a point, $O$, rays $OP$, $OQ$, $OR$ and $OS$ are drawn all in the same plane, and points $P$ and $R$ are on opposite sides of the line $SOQ$.
   If $\angle POQ = \angle ROQ$, prove that $\angle POS = \angle ROS$.

3 If one angle of a triangle is equal to the sum of the other two angles, prove that the triangle is right-angled.

4 In the diagram below, given that $AC = CB$ and $DC = CE$, prove that $AB \parallel DE$.

5 $ABC$ is a right-angled isosceles triangle with the right angle at $C$.
   $D$ and $E$ are points on $AB$ such that $\angle ACD = \angle BCE$.
   Prove that $\triangle CDE$ is isosceles.

6 In the diagram below, if $AB \parallel CD$ and $AE = AC$, prove that $\angle ACE = \angle ECD$.

7 In the diagram below, if $AB \parallel DC$, $AD = AC$ and $AB = BC$:
   a prove that $\angle ADC = \angle ACD = \angle CAB = \angle BCA$
   b prove that $\angle DAC = \angle ABC$.

8 The three angles of a triangle are in the ratio 3:5:7. Find the size of each angle.

9 $ABC$ is a triangle in which $AB = AC$.
   $AB$ is produced to $D$ so that $BD = BC$.
   Prove that $\angle ACB = 2 \angle DCB$.

10 $ABCD$ is a parallelogram and $O$ is the midpoint of $AB$.
   If $DC = 2AD$, prove that $\angle DOC = 90^\circ$. 
11. P, R and T are points on AB, and Q and S are points on AC, in the triangle ABC. If \( AP = PQ = QR = RS = ST \), prove that \( \angle CST = 5 \angle AQP \).

12. ABC is an equilateral triangle. K is a point on BC such that \( \angle CAK = 3 \angle KAB \). A line, KL, is drawn perpendicular to BC to meet AB at L. Prove that \( AL = KL \).

13. ABCDE is a regular pentagon. BD cuts CE at P. Prove that BP = BA.

14. ABC is a triangle. AP is the perpendicular from A to the bisector of \( \angle ABC \). If PQ is drawn parallel to BC to cut AB at Q, prove that \( AQ = QB = PQ \).

15. In the figure below, prove that \( x = 540 - (a + b + c) \)

16. The altitudes BD and CE of \( \triangle ABC \) meet at H. If HB = HC, prove that AB = AC.

17. ABCD is a parallelogram. BP and DQ are two parallel lines cutting AC at P and Q respectively. Prove that BPDQ is a parallelogram.

18. ABC is an isosceles triangle with \( AB = AC \). The bisector of \( \angle ABC \) meets AC at D. P is a point on AC produced so that \( \angle ABP = \angle ADB \). Prove that BC = CP.

19. ABC is an isosceles triangle with \( AB = AC \); BA is produced to E. The bisector of \( \angle ACB \) meets AB at D. If \( \angle ACD = x^\circ \), show that \( \angle CDA = 3x^\circ \) and \( \angle CAE = 4x^\circ \).

20. ABCD is a quadrilateral with \( \angle DAB = 90^\circ = \angle DCB \). The bisectors of \( \angle ADC \) and \( \angle ABC \) meet the diagonal AC at E and F respectively. Prove that DE and BF are parallel.

21. ABC is an isosceles triangle with \( AB = AC \); BC is produced to D so that CD = CA; CD is then produced to E so that DE = DA; EA is then produced to any point F. If \( \angle AEB = x^\circ \), show that \( \angle FAB = 5x^\circ \).
15.3 Congruent triangles

Two plane figures are congruent if they are equal in all respects. If one figure is placed on top of the other, with corresponding points touching, they will fit perfectly.

![Figure 15-10](image1)

In Figure 15-10, \( \triangle ABC \) and \( \triangle DEF \) are congruent. The three angles of one triangle have the same size as the three angles of the other, and the sides opposite these angles are equal in length, i.e.

\[
\angle A = \angle D \quad BC = EF \\
\angle B = \angle E \quad AC = DF \\
\angle C = \angle F \quad AB = DE
\]

\( \triangle ABC \cong \triangle DEF \), where \( \cong \) means 'is congruent to'.

The following four conditions for triangles show that all six corresponding parts of each triangle need not be given before we can say that the triangles are congruent.

In order to prove that two triangles are congruent, it can be shown that it is sufficient to prove that:

1. **two sides and the included angle** of one triangle are respectively equal to two sides and the included angle of the other triangle (SAS), as in Figure 15-11.

![Figure 15-11](image2)

2. **the three sides** of one triangle are respectively equal to the three sides of the other triangle (SSS), as in Figure 15-12.

![Figure 15-12](image3)

3. **two angles and a side** of one triangle are respectively equal to two angles and the corresponding side of the other triangle (ASA), as in Figure 15-13.
4 *the hypotenuse and one side* of a right-angled triangle are equal to the hypotenuse and the corresponding side of the other right-angled triangle *(RHS)*, as in Figure 15-14.

You will observe that two triangles are *not* necessarily congruent if:

a) the three angles of one triangle are equal to the three angles of the other triangle *(AAA)*;

Such triangles are *similar*.

b) two sides and an angle opposite one of these sides of one triangle are equal to two sides and the corresponding angle of the other triangle *(ASS)*.

Triangle 1 and Triangle 3 are congruent.
Triangle 1 and Triangle 2 are *not* congruent.

This situation is often referred to as the *ambiguous case* because sometimes it is possible to draw two different triangles, Triangle 1 and Triangle 2, that are not congruent.

Congruence properties are useful for proving certain geometrical facts.
Example 3
Prove that:

a  the base angles of an isosceles triangle are equal
b  the line drawn from the vertex of an isosceles triangle to the midpoint of the base is perpendicular to the base.

\[ \triangle ABC \text{ with } AB = AC \]

Given: An isosceles triangle \( \triangle ABC \) with \( AB = AC \)

To prove:

a  \( \angle ABC = \angle ACB \)

b  \( \angle APB = 90^\circ \)

Construction:

Join vertex \( A \) to midpoint \( P \) of the base.

Proof:

a  In \( \triangle ABP \) and \( \triangle ACP \):

\[ AP = AP \text{ (common to both triangles)} \]
\[ BP = CP \text{ (construction)} \]
\[ AB = AC \text{ (given)} \]

\[ \therefore \triangle ABP \equiv \triangle ACP \text{ (SSS)} \]

\[ \therefore \angle ABP = \angle ACP \]

b  It also follows from the congruent triangles that:

\[ \angle APB = \angle APC \]

But:

\[ \angle APB + \angle APC = 180^\circ \]

\[ \therefore \angle APB = 90^\circ \]

Example 4
Prove that, if two of the opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

\[ \text{Figure 15-18} \]
Given: $ABCD$ is a quadrilateral with $AB = DC$ and $AB \parallel DC$.

To prove:

$ABCD$ is a parallelogram.

Construction:

Draw the diagonal $AC$.

Proof: In $\triangle ABC$ and $\triangle ADC$:

$AC = AC$ (common)

$\angle BAC = \angle DCA$ ($AB \parallel DC$)

$AB = DC$ (given)

$\therefore \triangle ABC \cong \triangle ADC$ (SAS)

$\therefore \angle ACB = \angle DAC$

Since these are alternate angles, $BC \parallel AD$

$\therefore ABCD$ is a parallelogram (opposite sides are parallel).

**Example 5**

$ABCD$ is a square, $P$, $Q$ and $R$ are points on $AB$, $BC$ and $CD$ respectively, such that $AP = BQ = CR$.

Prove that:

a $PQ = QR$

b $\angle PQR = 90^\circ$

**Proof:** In $\triangle PBQ$ and $\triangle QCR$

$PB = QC$ (\therefore AP = BQ and $AB = BC$).

$BQ = CR$ (given)

$\angle PBQ = \angle QCR$ (right angles)

$\therefore \triangle PBQ \cong \triangle QCR$ (SAS)

a $\therefore PQ = QR$

b $\angle BPQ = \angle RQC$ (\therefore $\triangle$s are congruent)

But: $\angle BPQ + \angle BQP = 90^\circ$ (sum of angles of a $\triangle = 180^\circ$)

$\therefore \angle RQC + \angle BQP = 90^\circ$

$\therefore \angle PQR = 90^\circ$ (\therefore $\angle RQC + \angle BQP + \angle PQR = 180^\circ$)

**Note:**

\'\therefore\' means 'because'
Example 6
Squares $ABXY$ and $ACPQ$ are drawn outwardly on the sides $AB$ and $AC$ respectively of $\triangle ABC$ right-angled at $A$. Perpendiculars to $BC$ at $B$ and $C$ meet $XY$ and $PQ$, produced if necessary, at $M$ and $N$ respectively. Prove that:

a. $\triangle ABC$, $\triangle BXM$ and $\triangle CPN$ are congruent
b. $BCNM$ is a square.

**Proof:**

**a.** In $\triangle ABC$ and $\triangle BXM$:

\[
\begin{align*}
AB &= BX \text{ (sides of a square)} \\
\angle BAC &= \angle BXM \text{ (right angles)} \\
\angle ABC + \angle ABM &= 90^\circ = \angle XBM + \angle ABM \\
\therefore \angle ABC &= \angle XBM \\
\therefore \triangle ABC &\cong \triangle BXM \text{ (ASA)}
\end{align*}
\]

Similarly, in $\triangle ABC$ and $\triangle CPN$:

\[
\begin{align*}
AC &= CP \text{ (sides of a square)} \\
\angle BAC &= \angle CPN \text{ (right angles)} \\
\angle ACB + \angle ACN &= 90^\circ = \angle PCN + \angle ACN \\
\therefore \angle ACB &= \angle PCN \\
\therefore \triangle ABC &\cong \triangle CPN \text{ (ASA)}
\end{align*}
\]

**b.**

\[
\begin{align*}
\angle MBC &= \angle NCB \text{ (right angles)} \\
\therefore BM &= CN \\
BC &= BM (\triangle ABC \cong \triangle BXM) \\
BC &= CN (\triangle ABC \cong \triangle CPN) \\
\therefore BM &= CN \\
\therefore BCNM \text{ is a parallelogram (Example 4)}
\end{align*}
\]

However, since $\angle MBC$ is a right angle and $BC = BM$, the parallelogram is a square.
Exercises 15b

1. In the diagram below, $AB \parallel DC$ and $AO = OC$. Prove that $BO = OD$ by proving that $\triangle AOB$ and $\triangle COD$ are congruent.

2. $ABCD$ is a square and $CX = CY$. Prove that $AX = AY$, and $\angle AXB = \angle AYD$, by proving that $\triangle ABX$ and $\triangle ADY$ are congruent.

3. Two line segments, $AD$ and $BC$, bisect each other at $O$. Prove that $AB = CD$ and $AB \parallel CD$ by showing that $\triangle AOB$ and $\triangle COD$ are congruent.

4. Prove that, if the straight line drawn from the vertex of a triangle to the midpoint of the base is perpendicular to the base, the triangle is isosceles.

5. In the diagram below, given that $AB = AC$ and $DB = DC$, show that $\angle ABD = \angle ACD$. 
6 In the diagram below, if $AC = AD$, and $AB$ bisects $\angle CAD$, prove that:
   a $\triangle ABC \equiv \triangle ABD$
   b $BC = BD$.

7 $ABC$ is a triangle in which $\angle B = \angle C$. The bisector of angle $A$ meets $BC$ at $D$. Prove that $\triangle ABD$ and $\triangle ACD$ are congruent, and so prove that $AD$ is perpendicular to $BC$.

8 $ABCD$ is a square. $X$ and $Y$ are points on $BC$ and $CD$ respectively, such that $BX = CY$. Prove that:
   a $AX = BY$
   b $AX \perp BY$.

9 $P$ is a point inside a square, $ABCD$, such that triangle $PDC$ is equilateral. Prove that:
   a $\triangle APD \equiv \triangle BPC$
   b $\triangle APB$ is isosceles.

10 $P$ and $Q$ are the midpoints of the equal sides $AB$ and $AC$ of the isosceles triangle $ABC$. Prove that $PC = QB$.

11 $PQR$ is an isosceles triangle with $PQ = PR$. $SQR$ is also an isosceles triangle with $SQ = SR$, and $S$ is on the side of $QR$ opposite to $P$. Prove that:
   a $\triangle PQS \equiv \triangle PRS$
   b $P$, $X$ and $S$ are collinear, where $X$ is the midpoint of $QR$.

12 $ABCD$ is a quadrilateral with $AB = DC$ and $\angle BAC = \angle BDC$. Prove that:
   a $PB = PC$
   b $PA = PD$
   c $AC = BD$
   d $\triangle ABC \equiv \triangle DBC$. 
13 Prove that the opposite sides and angles of a parallelogram are equal.

14 From the midpoint, $O$, of $AB$, the line $CD$ is drawn at right angles. Prove that:
   a $\triangle AOC \cong \triangle BOC$
   b $\triangle AOD \cong \triangle BOD$
   c $\triangle CAD \cong \triangle CBD$

The quadrilateral $ACBD$ sometimes is called a 'kite'.

15 $ABC$ is an acute-angled triangle. The bisector of angle $A$ meets the perpendicular bisector of $BC$ at $P$.
   a Prove that $\triangle PBM \cong \triangle PCM$ and so prove that $PB = PC$.
   b Draw perpendicular lines $PX$ and $PY$ to $AB$ and $AC$ respectively.
      Prove that $\triangle APX \cong \triangle APY$ and so prove that $AX = AY$ and $PX = PY$.
   c Prove that $\triangle PXB \cong \triangle PYC$ and so prove that $XB = YC$.
   d Show that $AB = AC$.

You have now proved that any $\triangle ABC$ is isosceles. Where is the fallacy in the above proof?

16 $ABC$ is a triangle, right-angled at $A$. Squares $ACDE$ and $BCFG$ are drawn on $AC$ and $BC$ respectively. Prove that $AF = BD$.

17 $ABC$ is an isosceles triangle with $AB = AC$. $O$ is the midpoint of $BC$, and $OP$ and $OQ$ are drawn perpendicular to $AB$ and $AC$ respectively. Show that $OP = OQ$.

18 Prove that the diagonals of a rectangle are equal.

19 $E, F, G$ and $H$ are the midpoints of the sides $AB, BC, CD$ and $DA$ respectively of the parallelogram $ABCD$. Assuming that the opposite sides and angles of a parallelogram are equal, prove that:
   a $\triangle AEH \cong \triangle CFG$
   b $\triangle EBF \cong \triangle DHG$
   c $EFGH$ is a parallelogram.
20. P and Q are two points inside a parallelogram \(ABCD\), such that \(AP = QC\) and \(AP \parallel QC\). Prove that:
   a. \(\triangle APC \cong \triangle AQC\)
   b. \(PC \parallel AQ\)
   c. \(PC = AQ\)
   d. \(AQCP\) is a parallelogram.

21. \(ABCD\) is a parallelogram whose diagonals intersect at \(O\). Through \(O\), a straight line \(XY\) is drawn to meet \(AB\) and \(CD\) at \(X\) and \(Y\) respectively. Prove that \(OX = OY\).

22. \(O\) is a point inside an equilateral \(\triangle ABC\). \(OAP\) is an equilateral triangle, such that \(O\) and \(P\) are on opposite sides of \(AB\). Prove that \(BP = OC\).

23. \(ABCD\) is a parallelogram. \(ABXY\) and \(ADPQ\) are squares drawn outwards on \(AB\) and \(AD\) respectively. Prove that \(YQ = AC\).

15.4 Similar triangles

Two triangles are similar if:
   a. the angles are equal, i.e. the triangles are equiangular
   or b. the corresponding sides are proportional
   or c. two pairs of corresponding sides are proportional and their included angles are equal.

In the two triangles in Figure 15-21, the corresponding sides are:
   \[
   \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP},
   \]
   and the equal ratios are:
   \[
   \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}.
   \]

The tests for similarity of the triangles \(ABC\) and \(PQR\) are:
   a. \(\angle A = \angle P; \; \angle B = \angle Q; \; \angle C = \angle R\).
   b. \(\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}\)
   c. \(\frac{AB}{PQ} = \frac{AC}{PR}\) and \(\angle A = \angle P\).

We use the symbol \(\sim\) to denote 'is similar to'.
\[
\triangle ABC \sim \triangle PQR
\]

Example 7

If \(PQ \parallel BC\), prove that \(\triangle APQ \sim \triangle ABC\).
**Proof:** In $\triangle APQ$ and $\triangle ABC$:

$\angle APQ = \angle ABC$ (corresponding angles)

$\angle AQP = \angle ACB$ (corresponding angles)

$\angle QAP = \angle CAB$ (common angle)

∴ The triangles are equiangular and so $\triangle APQ \sim \triangle ABC$.

*Note:*

To prove that two triangles are equiangular, it is sufficient to prove that two angles of one triangle are equal to two angles of the other. Why?

Since the two triangles in Figure 15–22 overlap, it is convenient to draw them separately (see Figure 15–23).

![Figure 15–23](image)

The corresponding sides are in the same ratio:

$$\frac{AP}{AB} = \frac{AQ}{AC} = \frac{PQ}{BC} = \frac{3}{5}$$

The enlargement factor is $1 \frac{1}{3}$.

The sides of $\triangle APQ$ are $\frac{3}{5}$ the corresponding sides of $\triangle ABC$.

The sides of $\triangle ABC$ are $\frac{5}{3}$ the corresponding sides of $\triangle APQ$.

If $AQ = 3.6\text{ cm}$, $AC = \frac{5}{3} \times 3.6 = 6\text{ cm}$.

If $BC = 3.2\text{ cm}$, $PQ = \frac{3}{5} \times 3.2 = 1.92\text{ cm}$.

**Example 8**

![Figure 15–24](image)

Given: $\angle PAB = \angle BQC$

To prove: $\triangle APB \sim \triangle BCQ$
Proof:
\[ \angle PAB = \angle BQC \text{ (given)} \]
\[ \angle ABP = \angle CBQ \text{ (vertically opposite)} \]
\[ \therefore \angle APB = \angle BCQ \text{ (remaining angles are equal)} \]
\[ \therefore \triangle APB \sim \triangle BCQ \]

If \( AB = 4.2 \text{ cm} \) and \( BQ = 2.8 \text{ cm} \), the sides of \( \triangle BCQ \) are \( \frac{2}{3} \) of the corresponding sides of \( \triangle ABP \).

\[ \frac{BC}{BP} = \frac{BQ}{AB} = \frac{CQ}{AP} = \frac{2}{3} \]

or:

\[ \frac{BP}{BC} = \frac{AB}{BQ} = \frac{AP}{CQ} = \frac{3}{2} \]

If \( PB = 3 \text{ cm} \):

\[ BC = \frac{2}{3} \times 3 \text{ cm} = 2 \text{ cm} \]

If \( CQ = 2.4 \text{ cm} \):

\[ AP = \frac{3}{2} \times 2.4 \text{ cm} = 3.6 \text{ cm} \]

**Exercises 15c**

1 a If \( \angle APQ = \angle ABC \), prove \( \triangle APQ \sim \triangle ABC \).

2 If \( \angle S = \angle Q \):

a prove \( \triangle PST \sim \triangle RQT \)

b complete the ratios \( \frac{PS}{RQ} = \frac{PT}{?} = \frac{ST}{?} \).
3 a If $ST \parallel QR$:
   (i) prove $\triangle PST \sim \triangle PQR$
   (ii) complete the ratios $\frac{PS}{PQ} = \frac{?}{QR} = \frac{?}{PR}$

b If $QR = 5$ cm, $ST = 2$ cm, and $SP = 1$ cm, find the length of $PQ$.

4 $\triangle ABC$ and $\triangle ADE$ are right-angled.
   a Prove $\triangle ABC \sim \triangle AED$.
   b Complete the ratios $\frac{AB}{?} = \frac{AC}{?} = \frac{BC}{?}$.

5 $\triangle ABC$ has a right angle at $B$. The perpendicular $BD$ has been drawn.
   a Write three pairs of similar triangles you can find in the diagram.
   b Prove that:
      (i) $BD^2 = AD \cdot DC$
      (ii) $BC^2 = CA \cdot CD$
   c If $AD = 8$ cm and $BD = 6$ cm, write the lengths of $DC$, $AB$ and $BC$.

6 a A straight line through the vertex, $A$, of a parallelogram, $ABCD$, meets $BC$ at $E$ and $DC$ produced at $F$. Prove that:
   (i) $\triangle ABE \sim \triangle CEF$
   (ii) $\triangle ABE \sim \triangle ADF$
   b Complete the ratio $\frac{AB}{FD} = \frac{BE}{?}$

7 $PQR$ is a triangle, and a line drawn parallel to $PQ$ meets $QR$ at $X$ and $PR$ at $Y$. Through $P$, a line is drawn parallel to $QR$, meeting $XY$ produced at $Z$. Name three similar triangles.

8 $ABC$ is a triangle and $BE$ and $CF$ are drawn perpendicular to $AC$ and $AB$ respectively. Prove that triangles $AFC$ and $AEB$ are similar.
9 The sides $AB$ and $DC$ of a quadrilateral are produced to meet at $E$. If $\angle EBC = \angle ADE$, prove that $\triangle EBC \sim \triangle EDA$.

State two ratios each of which is equal to $\frac{BC}{DA}$.

10 In $\triangle ABC$, $D$, $E$ and $F$ are the midpoints of $BC$, $CA$ and $AB$ respectively. Prove that $\triangle ABC \sim \triangle DEF$. What is the ratio of their corresponding sides?

11 The diagram below (not to scale), illustrates the principle of a camera. Light from the object, $AB$, passes through a small hole, $P$, to form an inverted image, $DC$.

a Prove that $\triangle APB \sim \triangle DPC$.

b How far in front of $P$ must an object 2.7 m high be placed so that an image 2.5 cm high is formed 3 cm behind $P$?

12 $ABCD$ is a billiard table with dimensions of 12 units by 6 units. A player strikes a ball at $E$ and it hits a ball at $F$.

a Name two similar triangles in this situation.

b If $AE = 4$ units and $CF = 4$ units, what is the distance $BX$?

15.5 Theorem of Pythagoras

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

Given: $\triangle ABC$ with $\angle A = 90^\circ$
To prove: $BC^2 = AB^2 + AC^2$
Construction:
   Draw $AD \perp BC$
\textbf{Proof:} In $\triangle ABD$ and $\triangle ABC$:

\begin{align*}
\angle BDA &= \angle BAC \text{ (right angles)} \\
\angle DBA &= \angle ABC \text{ (common angle)} \\
\therefore \quad \angle BAD &= \angle ACB \text{ (third angles equal)} \\
\therefore \quad \triangle ABD &\sim \triangle ABC \\
\therefore \quad \frac{AB}{BD} &= \frac{BC}{AB}
\end{align*}

i.e. \quad \frac{AB}{BD} = \frac{BC}{AB}

\begin{align*}
\therefore \quad AB^2 &= BC \cdot BD \quad \text{ ........................................... (1)}
\end{align*}

In $\triangle ACD$ and $\triangle ABC$:

\begin{align*}
\angle ADC &= \angle BAC \text{ (right angles)} \\
\angle ACD &= \angle ACB \text{ (common angle)} \\
\therefore \quad \angle DAC &= \angle ABC \text{ (third angles equal)} \\
\therefore \quad \triangle ACD &\sim \triangle ABC \\
\therefore \quad \frac{AC}{AC} &= \frac{DC}{AC}
\end{align*}

i.e. \quad \frac{AC}{DC} = \frac{AC}{AC}

\begin{align*}
\therefore \quad AC^2 &= BC \cdot DC \quad \text{ ........................................... (2)}
\end{align*}

Add (1) and (2):

\begin{align*}
\therefore \quad AB^2 + AC^2 &= BC \cdot BD + BC \cdot DC \\
&= BC(BD + DC) \\
&= BC \cdot BC \\
&= BC^2
\end{align*}

\textbf{Note:}

$\triangle ABD$ and $\triangle ADC$ are also similar, and so $\frac{DA}{DC} = \frac{DB}{DA'}$ i.e. $DA^2 = DB \cdot DC$ \quad \text{ ............ (3)}

In words, (3) can be expressed as:

\textit{The square on the perpendicular is equal to the product of the segments of the base.}

In words (1) and (2) can be expressed as:

\textit{The square on either of the sides containing the right angle is equal to the product of the hypotenuse and the segment of the hypotenuse adjacent to that side.}

\textit{The converse} of a statement is formed by interchanging the hypothesis and conclusion or, in a geometrical proof, interchanging what is given and what has to be proved. Converse are not always true, e.g.

\textit{Statement:} If two triangles are congruent, then they are similar (True).

\textit{Converse:} If two triangles are similar, then they are congruent (False).

The converse of the Theorem of Pythagoras states:

\textit{If the square on a side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these sides must be a right angle.}
Given: \( \triangle ABC \) in which \( AC^2 = AB^2 + BC^2 \)

To prove: \( \angle ABC = 90^\circ \)

Construction: Draw \( \triangle DEF \) in which \( EF = BC \), \( DE = AB \) and \( \angle DEF = 90^\circ \)

Proof: In \( \triangle DEF \):
\[
FD^2 = DE^2 + EF^2 \quad \text{(Theorem of Pythagoras)}
\]
But:
\[
AC^2 = AB^2 + BC^2 \quad \text{(Given)}
\]
\[
\therefore \quad FD^2 = AC^2 \quad \text{(AB = DE, and BC = EF)}
\]
\[
\therefore \quad FD = AC
\]
In \( \triangle ABC \) and \( \triangle DEF \):
\[
AB = DE \quad \text{(Construction)}
\]
\[
BC = EF \quad \text{(Construction)}
\]
\[
AC = FD \quad \text{(Proved)}
\]
\[
\therefore \quad \triangle ABC = \triangle DEF \quad \text{(SSS)}
\]
\[
\therefore \quad \angle ABC = \angle DEF = 90^\circ \quad \text{(since \( \angle DEF = 90^\circ \))}
\]

Example 9

\( \triangle ABC \) is an equilateral triangle and \( X \) is a point on \( BC \) such that \( CX = 2BX \). Prove that \( AX^2 = 7BX^2 \).

![Figure 15-27](image)

Draw \( AP \perp BC \). \( P \) is the midpoint of \( BC \).

In \( \triangle AXP \):
\[
AX^2 = AP^2 + PX^2 = AB^2 - BP^2 + PX^2 \quad \text{................................. (1)}
\]
Since \( CX = 2BX \), then \( BC = 3BX \) and so \( BC^2 = 9BX^2 \)
\[
AB^2 = BC^2 = 9BX^2
\]
Since \( BP = \frac{1}{2}BC \), then \( BP^2 = \frac{1}{4}BC^2 = \frac{9}{4}BX^2 \)
Since \( PX = \frac{1}{2}BX \), then \( PX^2 = \frac{1}{4}BX^2 \)
So, from (1):
\[
AX^2 = 9BX^2 - \frac{9}{4}BX^2 + \frac{1}{4}BX^2 = 7BX^2
\]

Exercises 15d
1. Calculate the length of each side of a rhombus whose diagonals are:
   a. 12 cm, 16 cm
   b. 4 cm, 9.6 cm
2. The lengths of the sides of a rhombus and the length of one diagonal are respectively 20 cm and 24 cm. Calculate the length of the other diagonal.
A motorist starts from a point, \( A \), and travels 32 km North, 24 km East, 25 km South and finally returns to \( A \). What is the total distance travelled?

Two roads intersect at right angles. Two cars start at the same time from the intersection. One travels at 60 km/h along one road and the other travels at 80 km/h along the other road. How far apart are they after:

\( \text{a} \) 6 minutes?  
\( \text{b} \) 15 minutes?  
\( \text{c} \) 30 minutes?  
\( \text{d} \) 36 minutes?  
\( \text{e} \) 45 minutes?  
\( \text{f} \) 60 minutes?

In Figure \( \text{a} \) below, \( AB = 5 \text{ cm}, \ BC = 6 \text{ cm}, \ DE = 10 \text{ cm} \) and \( AD = 13 \text{ cm} \). Find the length of \( CD \).

In Figure \( \text{b} \) above, \( AB = 24 \text{ cm} \) and \( AC = CD = 25 \text{ cm} \). Find the length of:

\( \text{a} \) \( BC \)  
\( \text{b} \) \( AD \)

The two equal sides of an isosceles triangle are each 13 cm long and the base length is 10 cm. Calculate the height of the triangle.

Calculate the length of the altitude of the isosceles triangles whose sidelongths are:

\( \text{a} \) 10 cm, 10 cm, 12 cm  
\( \text{b} \) 25 cm, 25 cm, 14 cm  
\( \text{c} \) 5.2 cm, 5.2 cm, 4 cm  
\( \text{d} \) 3 cm, 3 cm, 3.6 cm

A rectangular sheet of paper is 24 cm by 18 cm. If it is folded flat along the line \( CD \), how many centimetres is \( B \) then closer to \( A \)?

When a rectangular sheet of paper 16 cm by 9 cm is cut in the manner shown in the diagram below, the pieces can be rearranged to form a square. What is the perimeter of this square?
Two vertical poles standing on horizontal ground at points 9 m apart are of lengths 6 m and 12 m. Find the length of the straight wire joining the tops of the poles.

12 \(ABC\) is a triangle in which \(AB\) is greater than \(AC\). \(AD\) is drawn perpendicular to \(BC\). Prove that:

\[AB^2 + DC^2 = AC^2 + DB^2\]

13 \(ABCD\) is a quadrilateral whose diagonals intersect at right angles. Prove that:

\[AB^2 + CD^2 = BC^2 + DA^2\]

14 In a trapezium \(ABCD\), \(AB\) is parallel to \(DC\) and the diagonal \(BD\) is perpendicular to \(AB\). Prove that:

\[AB^2 + BC^2 = AD^2 + DC^2\]

15 \(ABC\) is a triangle in which \(AB = 3\) cm, \(BC = 5\) cm and \(\angle A = 90^\circ\). Calculate the length of the perpendicular from \(A\) to \(BC\).

16 \(PQR\) is a triangle, right-angled at \(P\). \(S\) is any point on \(PQ\) and \(T\) is any point on \(PR\). Prove that:

\[QT^2 + RS^2 = QR^2 + ST^2\]

17 In the parallelogram \(ABCD\), the diagonal \(BD\) is perpendicular to \(AB\). If the diagonals intersect at \(E\), prove that:

\[AD^2 = AE^2 + 3ED^2\]

18 \(AD\) is an altitude of \(\triangle ABC\). If \(BD = x\) cm, \(AD = 2x\) cm and \(DC = 4x\) cm, prove that \(\angle BAC = 90^\circ\).

19 \(ABC\) is an equilateral triangle, \(BC\) is produced to \(E\) so that \(CE = \frac{1}{2} BC\). If \(CE = x\) units, prove that \(AE = \sqrt{7}x\).

20 \(ABC\) is a right-angled triangle with the right angle at \(A\). If \(P\), \(Q\) and \(R\) are the midpoints of \(BC\), \(CA\) and \(AB\) respectively, prove that \(BQ^2 + CR^2 = 5AP^2\).
15.6 Circle theorems

Theorem 3

The angle that an arc of a circle subtends at the centre is twice the angle it subtends at the circumference.

Figure 15–28

Given: $AB$ is an arc of a circle, centre $O$, and $P$ is any point on the circle but not on the arc $AB$.

To prove:

$$\angle AOB = 2 \angle APB$$

Construction:

Draw the ray $POQ$.

Proof:

In the isosceles triangle $APO$, $PO$ is produced to $Q$.

$$\therefore x = a + a \quad \text{(Exterior angle is equal to the sum of the two remote interior angles)}$$

In the isosceles triangle $BPO$, $PO$ is produced to $Q$.

$$\therefore y = b + b \quad \text{(Exterior angle is equal to the sum of the two remote interior angles)}$$

$$\therefore x + y = 2a + 2b = 2(a + b)$$

$$\therefore \angle AOB = 2 \angle APB$$

As an exercise, prove the theorem when the point $P$ is located as shown in the following two circles (Figures 15–29 and 15–30).
Theorem 4
An angle in a semicircle is a right angle.

Given: Semicircle $APB$.
To prove:
$\angle APB = 90^\circ$

Proof:
$\angle AOB = 2 \angle APB$ (Angle at the centre is twice the angle at the circumference)
But: $\angle AOB = 180^\circ$ (A straight angle)

$\therefore \angle APB = 90^\circ$

Theorem 5
Angles in the same segment are equal.
or
Angles subtended at the circumference by the same arc are equal.

Given: $\angle APB$ and $\angle AQB$ are any two angles in the major segment $APQB$, or are any two angles subtended at the circumference by the minor arc $AB$.
To prove:
$\angle APB = \angle AQB$

Proof:
$\angle AOB = 2 \angle APB$ (Angle at centre is double the angle at the circumference)
$\angle AOB = 2 \angle AQB$ (Same reason)

$\therefore \angle APB = \angle AQB$
Example 10
Given: $AC$ is a diameter, $\angle ABD = 62^\circ$ and $\angle ACB = 54^\circ$, find $x$, $y$, and $z$.

![Figure 15-34]

\[ \angle ABC = 90^\circ \quad \text{(Angle in a semicircle is a right angle)} \]
\[ x + 62 = 90 \]
\[ x = 28 \]
\[ \angle ACD = \angle ABD \quad \text{(Angles in same segment are equal)} \]
\[ y = 62 \]
\[ z = 180 - (90 + 54) \]
\[ z = 36 \quad \text{(Sum of angles of } \triangle ABC = 180^\circ) \]

Example 11
$A$, $B$, and $C$ are three points on the circumference of a circle such that chord $AB =$ chord $AC$. $P$ is any point on the circle, such that $A$ and $P$ are on opposite sides of $BC$. Prove that $\angle APB = \angle APC$.

Proof:
\[ \angle ABC = \angle ACB \quad (\therefore AB = AC) \]
\[ \angle ABC = \angle APC \quad \text{(angles in the same segment)} \]
\[ \angle ACB = \angle APB \quad \text{(angles in the same segment)} \]
\[ \therefore \]
\[ \angle APB = \angle APC \quad (\therefore \angle ABC = \angle ACB) \]

![Figure 15-35]

Note:
\[ \therefore \] means ‘because’.
Exercises 15e

1 In each of the following, find the value of the pronumerals. Give reasons for your answers. \((O\) is the centre of the circle.\)

\[ a \]
\[
\begin{array}{c}
O \\
40^\circ \\
b^\circ \\
a^\circ \\
\end{array}
\]

\[ b \]
\[
\begin{array}{c}
O \\
50^\circ \\
a^\circ \\
105^\circ \\
\end{array}
\]

\[ c \]
\[
\begin{array}{c}
O \\
x^\circ \\
62^\circ \\
b^\circ \\
\end{array}
\]

\[ d \]
\[
\begin{array}{c}
O \\
x^\circ \\
y^\circ \\
z^\circ \\
\end{array}
\]

\[ e \]
\[
\begin{array}{c}
O \\
x^\circ \\
y^\circ \\
z^\circ \\
\end{array}
\]

\[ f \]
\[
\begin{array}{c}
O \\
x^\circ \\
y^\circ \\
z^\circ \\
\end{array}
\]

2 Two chords, \(AB\) and \(CD\), of a circle intersect at a point, \(E\), inside the circle. Prove that triangles \(AEC\) and \(EDB\) are equiangular.

3 Two circles intersect at \(A\) and \(B\). \(AX\) is a diameter of one circle and \(AY\) is a diameter of the other. Prove that the points \(X\), \(B\) and \(Y\) are collinear, i.e. are on the same straight line. \((\text{Construction: Join the common chord } AB.\)\)

4 \(ABC\) is a triangle inscribed in a circle, centre \(O\), and \(AD\) is drawn perpendicular to \(BC\). Prove that \(\angle BAE = \angle DAC\)

5 \(AB\) is a chord of a circle, centre \(O\). The bisector of angle \(OAB\) meets the circle at \(D\). Prove that \(OD\parallel AB\).

6 \(A\), \(B\), \(C\) and \(D\) are four points in order on a circle. Chords \(AC\) and \(BD\) intersect at \(E\). If \(AD\parallel BC\), prove that triangles \(AED\) and \(BEC\) are isosceles.

7 \(ABC\) is an isosceles triangle inscribed in a circle, centre \(O\). \(AD\) is a diameter. Prove that \(\angle BDE = 90^\circ\).
8. A chord, \( BC \), and the diameter from \( A \) meet at a point, \( P \), such that \( OC = CP \). If \( \angle CPO = a^\circ \), prove that \( \angle AOB = 3a^\circ \). This is a method of trisecting an angle by construction.

![Diagram](image)

9. \( AB \) and \( AC \) are equal chords of a circle. \( AD \) and \( BE \) are parallel chords through \( A \) and \( B \) respectively. Prove that \( AE \) is parallel to \( CD \).

10. \( ABCD \) is a parallelogram. (See the diagram below.) Prove that \( \triangle ABE \) is isosceles.

![Diagram](image)

11. Two chords \( AB \) and \( CD \) intersect at point \( P \) outside the circle. Prove that \( \triangle APD \sim \triangle CPB \).

![Diagram](image)

12. \( AB \) and \( DC \) are equal chords of a circle. Prove that:
   a. \( \triangle CDE \) and \( \triangle BAE \) are congruent
   b. \( \triangle EDA \) is isosceles
   c. \( \triangle ADC \) and \( \triangle DAB \) are congruent
   d. \( AC \) and \( BD \) are equal chords.

![Diagram](image)

13. \( AB \) is a chord of a circle, centre \( O \). The circle with \( AO \) as diameter cuts \( AB \) at \( C \). Prove that \( C \) is the midpoint of \( AB \).
14 In a triangle $APB$, the side $AB$ remains fixed and the vertex $P$ moves in a plane, what is the locus of $P$ if angle $APB$ remains the same size?

15 In a triangle $APB$, angle $APB$ is a right angle. What is the locus of $P$ in a plane?

16 $A$, $B$, $C$ and $D$ are four points in order on a circle, centre $O$. If $\angle BAC = 28^\circ$ and $\angle OBA = 53^\circ$, calculate the angles $OAC$, $ACB$ and $ADC$.

17 Two perpendicular chords $AB$ and $CD$ of a circle, centre $O$, intersect at $E$. Prove that $\angle AOD + \angle BOC = 180^\circ$.

### 15.7 Cyclic quadrilaterals

A cyclic quadrilateral is a quadrilateral whose four vertices lie on the circumference of a given circle.

**Theorem 6**

a The opposite angles of a cyclic quadrilateral are supplementary.

b If a side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle.

![Figure 15–36](image)

Given: $ABCD$ is a cyclic quadrilateral

a To prove:

\[
\begin{align*}
b + d &= 180 \\
a + c &= 180
\end{align*}
\]

Construction: Join centre $O$ to $A$ and $C$.

**Proof:**

\[
\begin{align*}
x &= 2d \quad \text{(Angle at centre is double the angle at the circumference)} \\
y &= 2b \quad \text{(Same reason)}
\end{align*}
\]

\[
\begin{align*}
x + y &= 2(b + d) \\
\therefore \quad x + y &= 360 \\
\therefore \quad b + d &= 180
\end{align*}
\]

Similarly, by joining centre $O$ to $B$ and $D$ we can prove that:

\[
a + c = 180
\]

b To prove: $\angle CBE = \angle ADC$

**Proof:**

\[
\begin{align*}
\angle CBE + \angle CBA &= 180^\circ \quad \text{(straight angle)} \\
\angle ADC + \angle CBA &= 180^\circ \quad \text{(opposite angles supplementary)}
\end{align*}
\]

\[
\therefore \quad \angle CBE = \angle ADC
\]
**Example 12**

![Diagrams showing angles](image)

**Theorem 7 (Converse of Theorem 6)**

*If the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.*

![Diagram of quadrilateral with circle](image)

**Given:** \(ABCD\) is a quadrilateral with:

\[\angle DAB + \angle DCB = 180^\circ\]

**To prove:** \(ABCD\) is cyclic

**Construction:**

Suppose the circle through \(B, A\) and \(D\) does not pass through \(C\); let it cut \(DC\) (produced if necessary) at \(E\). Join \(EB\).

**Proof:**

\[\angle DAB + \angle DEB = 180^\circ\] (opposite angles supplementary)

\[\angle DAB + \angle DCB = 180^\circ\] (given)

\[\therefore \angle DEB = \angle DCB\]

But this is impossible, since an exterior angle of a triangle cannot be equal to an interior opposite angle.

\[\therefore \text{ } E \text{ must coincide with } C.\]

So, the circle through \(D, A\) and \(B\) must pass through \(D\).

i.e. \(ABCD\) is cyclic.

This converse provides us with the test for four concyclic points:

*Four points \(A, B, C\) and \(D\) in order are concyclic if the opposite angles of the quadrilateral \(ABCD\) are supplementary.*
Example 13
Given: $\angle DAC = 45^\circ$, $\angle BAC = 35^\circ$ and $\angle ABC = 85^\circ$, find $x$ and $y$.

![Figure 15-39]

$\angle DAB = 80^\circ (45^\circ + 35^\circ)$
$\angle DCB = 100^\circ$ (opposite angles supplementary)
$\angle ACB = 60^\circ$ (sum of angles of $\triangle ABC = 180^\circ$)

$\therefore \angle DCA = 40^\circ$ ($\angle DCB - \angle ACB$)

$\therefore x = 40$

$y = 95$ (opposite angles $B$ and $D$ are supplementary)

Example 14
Two circles intersect at $A$ and $B$. $PAQ$ and $XBY$ are straight lines through $A$ and $B$, cutting the circles at $P$ and $X$, and $Q$ and $Y$ respectively. Prove that $PX$ is parallel to $QY$.

![Figure 15-40]

Construction: Join $AB$; produce $AP$ to $R$

Proof: Quadrilaterals $PABX$ and $QABY$ are cyclic.

$\angle ABX = \angle RPX$ (exterior angle equals interior opposite)
$\angle ABX = \angle AQY$ (exterior angle equals the interior opposite angle)

$\therefore \angle RPX = \angle AQY$

$\therefore PX \parallel QY$ (corresponding angles equal)

Example 15
$ABCD$ is a cyclic quadrilateral and $AB$ is produced to $E$. The bisector of angle $ADC$ cuts the given circle around $ABCD$ at $F$. Prove that $FB$ (produced if necessary) bisects the angle $EBC$. 
Proof: \( \angle FBC = \angle CDF \) (angles in same segment)
\( \angle EBF = \angle ADF \) (exterior angle of cyclic quadrilateral \( FBAD \) equals interior opposite angle)

But: \( \angle ADF = \angle CDF \) (given)
\[ \therefore \] \( \angle FBC = \angle EBF \)

**Exercises 15f**

1. Find the value of the pronumerals in each of the following. (\( O \) is the centre of the circle.)

   ![Diagram](image)

   ```
   a
   b
   c
   d
   e
   f
   ```

2. \( ABC \) is an isosceles triangle with \( AB = AC \). \( P \) and \( Q \) are points on \( AB \) and \( AC \) respectively, and \( PQ \) is parallel to \( BC \). Prove that the quadrilateral \( PBCQ \) is cyclic.

3. Two circles intersect at \( A \) and \( B \). \( PAQ \) and \( XBY \) are parallel straight lines through \( A \) and \( B \), cutting the circles at \( P \) and \( Q \), and \( X \) and \( Y \) respectively. Join \( AB \) and prove \( \angle PXY = \angle PQY \).

4. \( ABCD \) is a cyclic quadrilateral with \( AB = AD \). If \( \angle ABD = x^\circ \), prove that \( \angle DCB = 2x^\circ \).

5. Prove that a parallelogram which is cyclic must be a rectangle.

6. \( ABC \) is an isosceles triangle with \( AB = AC \). A straight line through \( C \) perpendicular to \( AC \) meets the bisector of \( \angle BAC \) at \( D \). Prove that the points \( A, B, D \) and \( C \) are concyclic.
7 $AB$ and $AC$ are two chords of a circle on opposite sides of centre, $O$. $P$ and $Q$ are the midpoints of $AB$ and $AC$ respectively. Prove that $A$, $P$, $O$ and $Q$ are concyclic.

8 $ABCD$ is a cyclic quadrilateral in which $AD = DC = CB$. Prove that:
   a $\angle ADC = \angle DCB$
   b $DC \parallel AB$

9 $C$ is any point on a circle diameter $AB$. $P$ and $Q$ are points on the minor arcs $AC$ and $BC$. Prove that $\angle APC + \angle CQB = 3$ right angles.

10 $ABCD$ is a cyclic quadrilateral with $AB \parallel DC$. Prove that $\angle DAB = \angle ABC$.

11 $A$, $B$, $C$ and $D$ are four points in order on a circle. If $AB$ is a diameter and $\angle ADC = \angle BCD$, prove that:
   a $DC \parallel AB$
   b $\angle DBA = \angle CAB$

12 $A$, $B$ and $C$ are three points on a circle and $\triangle ABC$ is acute-angled. $AD$ is drawn parallel to $BC$, and $CD$ is drawn parallel to $BA$. Prove that $D$ cannot lie on the circle. ($D$ is the point of intersection of the two parallels).

13 $ABCD$ is a rectangle. The line through $C$ perpendicular to $AC$ meets $AB$ and $AD$ produced at $P$ and $Q$. Prove that $P$, $B$, $D$ and $Q$ are concyclic.

15.8 Tangents to a circle

Figure 15-42 shows a series of parallel secants $AB$. Observe that, as they move farther away from the centre of the circle, the points, $A$ and $B$ draw closer together until, finally, the two points $A$ and $B$ take up the same position. Let us call this point $P$. The secant in this position is called a tangent to the circle and $P$ is called the point of contact.

Definition

A tangent to a circle is a straight line that touches the circle in one point only.

Figure 15-43 shows that, arguing from symmetry, a tangent and the radius at the contact point are at right angles.

Axiom

A tangent to a circle is perpendicular to the radius drawn at the point of contact.
**Theorem 8**

The tangents to a circle from an external point are equal in length.

![Figure 15-44](image)

Given: $PQ$ and $PR$ are tangents to the circle with centre $O$.

To prove: $PQ = PR$

Construction:

Join $OQ$, $OP$ and $OR$

Proof: In $\triangle POQ$ and $\triangle POR$:

1. $OQ = OR$ (equal radii)
2. $OP = OP$ (common)
3. $\angle QOR = \angle POR$ (right angles)

$\therefore \triangle POQ \cong \triangle POR$ (RHS)

$\therefore PQ = PR$

Also: $\angle OPQ = \angle OPR$

and: $\angle POQ = \angle POR$

**Theorem 9**

If two circles touch, the line joining their centres passes through their point of contact.

![Figure 15-45](image)  
![Figure 15-46](image)

Given: $A$ and $B$ are the centres of two circles which touch at $P$ (externally in Figure 15-45 and internally in Figure 15-46).

To prove: $A$, $B$ and $P$ are collinear.

Proof: Since $XY$ is a tangent to circle, centre $A$, $\angle XPA$ = a right angle.

Since $XY$ is a tangent to circle, centre $B$, $\angle XPB$ = a right angle.

$\therefore PA$ and $PB$ are both perpendicular to $XY$.

$\therefore A$, $B$ and $P$ must be in a straight line.
15.9 Alternate segment

$XY$ is a tangent to the circle at $P$. The chord $PQ$ divides the circle into two segments (sections) $PQR$ and $PQS$.

The segment $PQR$ is said to be alternate to $\angle QPY$.

The segment $PQS$ is said to be alternate to $\angle QPX$.

![Figure 15-47](image1)

**Theorem 10 (Alternate segment theorem)**

The angles between a tangent and a chord through the point of contact are equal to the angles in the alternate segments.

![Figure 15-48](image2)

**Given:** $XY$ is a tangent at $P$ to the circle, centre $O$, and $PQ$ is a chord through $P$.

**To prove:**

a. $\angle QPY = \text{any angle in alternate segment } PTQ$

b. $\angle QPX = \text{any angle in alternate segment } PSQ$

**Construction:** Draw the diameter $PR$ and join $RQ$.

**Proof:**

a. $\angle QPY + \angle QPR = 90^\circ$ (tangent $\perp$ to radius)

$\angle PRQ + \angle QPR = 90^\circ$ (\(\angle PQR = 90^\circ\), angle in a semicircle)

$\therefore \quad \angle QPY = \angle PRQ$

$\therefore \quad \angle QPY = \text{any angle in segment } PTQ$

b. $\angle QPX + \angle QPY = 180^\circ$ (straight angle)

$\angle PSQ + \angle PRQ = 180^\circ$ ($PSQR$ is cyclic)

But: $\angle QPY = \angle PRQ$ (proved in a)

$\therefore \quad \angle QPX = \angle PSQ$

$\therefore \quad \angle QPX = \text{any angle in segment } PSQ$
Example 16

Two points, $A$ and $B$, are taken on a circle and $C$ is the other end of the diameter through $A$. If $AE$ is the perpendicular from $A$ on to the tangent at $B$, prove that $AB$ bisects angle $CAE$.

To prove:

\[ \angle BAE = \angle BAC \]

Proof:

\[ \angle ABC = 90^\circ \quad \text{(angle in a semicircle)} \]

\[ \angle EBA = \angle ACB \quad \text{(angles in the alternate segment)} \]

In $\triangle BAE$ and $\triangle ABC$:

\[ \angle AEB = \angle ABC \quad \text{(right angles)} \]

\[ \angle EBA = \angle ACB \quad \text{(angles in the alternate segment)} \]

\[ \therefore \quad \angle BAE = \angle BAC \quad \text{(remaining angles in the triangles)} \]

Exercises 15g

1 Find the value of the pronumerals in each of the following. In each case, $O$ is the centre of the circle.

2 Use the fact that two tangents, drawn to a circle from an external point, are equal in length to find the perimeter of $\triangle ABC$. 
3 In the right-angled triangle \(ABC\), \(BC = 24\) cm, \(DB = 3\) cm, \(AD = x\) cm. Use the Theorem of Pythagoras to form an equation in \(x\) and then solve it.

4 Calculate:
   a the distance of a chord of length 24 cm from the centre of a circle of radius 13 cm.
   b the length of the tangents drawn from an external point to the extremities of the chord.

5 Two circles of radii 5 cm and 8 cm touch each other externally. Calculate the length of the common tangent.

6 If the radii of two intersecting circles are 17 cm and 10 cm and the length of the common chord is 16 cm, calculate:
   a the length of the line joining the centres of the circles
   b the length of the common tangent.

7 If the radii of two intersecting circles are 51 cm and 74 cm and the length of the common chord is 48 cm, calculate:
   a the length of the line joining their centres
   b the length of the common tangent.

8 \(ABCD\) is a cyclic quadrilateral and \(FAE\) is a tangent at \(A\). Angle \(DAE = 50^\circ\). Calculate the size of:
   a \(\angle BAF\)
   b \(\angle BCD\)
   given also that \(BD \parallel FE\).

9 From a point, \(P\), outside a circle two straight lines are drawn. The first, \(PAB\), cuts the circle at \(A\) and \(B\). The second, \(PTX\), is a tangent to the circle at \(T\). \(AT\) and \(BT\) are joined. Prove that \(\angle BTX = \angle APT + \angle ATP\).

10 Two circles intersect at \(A\) and \(B\). The tangent to the first circle at \(A\) cuts the second circle at \(C\), and the tangent to the second circle at \(A\) cuts the first circle at \(D\). Prove that \(\triangle ABC\) and \(\triangle DBA\) are similar.

11 The sides of the quadrilateral \(PQRS\) are tangents to the circle. Calculate the size of each angle of the quadrilateral \(ABCD\).
12 PQ is a tangent at Q and PS\parallel QR. Prove that \angle SPQ = \angle QSR.

13 Two circles intersect at A and B. The tangent to the second circle at A cuts the first circle at C and the tangent to the first circle at B cuts the second circle at D. Prove that AD is parallel to BC.

14 Two circles touch internally at P. The lines PAB and PCD cut the smaller circle at A and C and the larger circle at B and D. Prove that, by drawing the common tangent at P, AC is parallel to BD.

15 Two circles touch internally at A. The tangent at P on the smaller circle cuts the larger circle at Q and R. Prove that AP bisects \angle RAQ.

16 The diagram shows three circles each of radius 2 cm. Calculate the height of the highest point, P, above AB.

17 Two circles, centres O and P, touch externally at A. The direct common tangent touches the circles at X and Y respectively. The common tangent at A meets the direct common tangent at Z. Prove that:
   a) ZX = ZY
   b) \angle OZP = 90°

18 Two circles intersect in X and Y. The tangent at X to the first circle cuts the second circle at A, and AY produced cuts the first circle at B. Prove that XB is parallel to the tangent at A to the second circle.

19 ABC is a triangle inscribed in a circle. The tangent at C meets AB produced at P, and the bisector of \angle ACB meets AB at Q. Prove that PC = PQ.

20 PA and PB are two tangents to a circle and X is the midpoint of the minor arc AB. Prove that:
   a)XA bisects \angle PAB
   b)XB bisects \angle PBA.
21. \(AB\) and \(CD\) are parallel chords of a circle. The tangent at \(B\) meets \(CD\) produced at \(E\). Prove that \(\triangle ABC\) and \(\triangle DBE\) are similar.

22. Two circles intersect at \(A\) and \(B\). A line through \(A\) cuts the first circle at \(P\) and the second circle at \(Q\). From an external point, \(T\), a tangent \(TP\) is drawn and \(TQ\) produced meets the second circle again at \(R\). Prove that the points \(P, T, R\) and \(B\) are concyclic.

23. Six equal circular discs are placed so that their centres lie on the circumference of a given circle and each disc touches its two neighbours. If the radius of the given circle is \(r\), find:
   a. the radius of a seventh disc which will touch each of the six
   b. the length of the outside perimeter of the figure.

### 15.10 Intersecting chords of a circle

**Theorem 11**

*If two chords intersect (either inside a circle or, when produced, outside the circle) the product of the two sections of one chord is equal to the product of the two sections of the other chord.*

**Given:** \(AB\) and \(CD\) are two chords of a circle intersecting at \(E\) inside the circle (Figure 15-50) and, when produced, outside the circle (Figure 15-51).

**To prove:**

\[AE \cdot EB = CE \cdot ED\]

**Construction:**
Join \(AC\) and \(BD\)

**Proof:** In \(\triangle AEC\) and \(\triangle BED\):

- \(\angle AEC = \angle BED\) (vertically opposite, Figure 15-50)
- \(\angle CAE = \angle CDB\) (common to both triangles, Figure 15-51)
- \(\angle CAE = \angle CDB\) (angles in same segment, Figure 15-50)
- \(\angle CAE = \angle CDB\) (exterior angle of cyclic quadrilateral = interior opposite angle, Figure 15-51)

\[\therefore \triangle AEC\text{ and }\triangle BED\text{ are equiangular and, therefore, similar.}\]

\[\therefore \frac{AE}{ED} = \frac{CE}{EB}\]

\[\therefore AE \cdot EB = CE \cdot ED\]
Theorem 12
If a chord, when produced, meets a tangent, the square on the tangent is equal to the product of the two sections of the chord.

Given: A chord, $AB$, and a tangent at $P$ intersecting at the point $E$.
To prove: $PE^2 = AE \cdot EB$

Construction:
Join $PA$ and $PB$

Proof: In $\triangle BPE$ and $\triangle APE$:
\[ \angle BPE = \angle PAE \] (alternate segment theorem)
\[ \angle BEP = \angle AEP \] (common angle)
\[ \therefore \triangle BPE \text{ and } \triangle APE \text{ are equiangular and, therefore, similar.} \]
\[ \therefore \frac{PE}{AE} = \frac{EB}{PE} \]
\[ \therefore PE^2 = AE \cdot EB \]

Exercises 15h

1 Two chords, $AB$ and $CD$, of a circle intersect at a point, $E$, inside the circle.
   a If $AE = 2 \text{ cm}$, $EB = 6 \text{ cm}$ and $CE = 3 \text{ cm}$, calculate $ED$.
   b If $AE = 5 \text{ cm}$, $AB = 9 \text{ cm}$ and $CD = 12 \text{ cm}$, calculate $CE$ and $ED$.

2 Two chords $PQ$ and $RS$ meet, when produced, at a point, $T$, outside the circle.
   a If $PQ = 5 \text{ cm}$, $QT = 3 \text{ cm}$ and $ST = 4 \text{ cm}$, calculate $RS$.
   b If $PT = 9 \text{ cm}$, $PQ = 5 \text{ cm}$ and $RS = 9 \text{ cm}$, calculate $ST$.

3 From a point, $P$, outside a circle a line, $PQR$, is drawn to meet the circle at $Q$ and $R$ and a tangent, $PT$, is drawn.
   a If $PQ = 2 \text{ cm}$ and $QR = 6 \text{ cm}$, calculate $PT$.
   b If $PT = 6 \text{ cm}$ and $RQ = 5 \text{ cm}$, calculate $PQ$.

4 From a point, $P$, outside a circle, two equal lines, $PQR$ and $PST$, are drawn meeting the circle at $Q$, $R$, $S$ and $T$. Prove that the chords $QR$ and $ST$ are equal.

5 If two chords, $AB$ and $CD$, of a circle bisect each other, prove that the chords are diameters of the circle.

6 $AB$ is a diameter and $AC$ is a tangent. Prove that:
   a $AC^2 = BC \cdot DC$
   b $AB^2 = BC \cdot BD$
   c $AB^2 + AC^2 = BC^2$
7 The common chord $BA$ produced meets the common tangent $PQ$ at $R$. Prove that $PR = QR$.

8 $AB$ is a diameter of a circle and $CD$ is a chord drawn perpendicular to $AB$, cutting it at $X$. Prove that $CX^2 = AX \cdot XB$.

9 The common chord, $AB$, of two intersecting circles is produced to any point, $P$. Prove that the tangents from $P$ to the two circles are equal.

**15.11 Concurrency theorems**

Two or more lines are ‘concurrent’ if they pass through the same point. We shall use the assumptions and the theorems appearing in the previous sections as well as the following two theorems.

**Theorem 13**

*Any point, $P$, on the perpendicular bisector of a line, $AB$, is equidistant from $A$ and $B.*

![Figure 15-53]

To prove: $PA = PB$

**Proof:** In $\triangle APX$ and $\triangle BPX$:

- $AX = BX$ (given)
- $\angle AXP = \angle BXP$ (right angles)

$PX$ is common.

$\therefore \triangle APX \cong \triangle BPX$ (SAS)

$\therefore PA = PB$

We shall assume the converse of this theorem to be true:

*Any point $P$, which is equidistant from $A$ and $B$, lies on the perpendicular bisector of the line $AB.*
**Theorem 14**

*Any point, P, on the bisector of angle BAC is equidistant from the two arms forming the angle BAC.*

To prove: \( PX = PY \)

**Proof:** In \( \triangle PAX \) and \( \triangle PAY \):
- \( \angle PAX = \angle PAY \) (given)
- \( PA \) is common.
- \( \angle PXA = \angle PYA \) (right angles)

\[ \therefore \triangle PAX \cong \triangle PAY \text{(AAS)} \]

\[ \therefore PX = PY \]

We shall assume the converse of this theorem to be true:

*Any point P, which is equidistant from the two arms forming the angle BAC, lies on the bisector of the angle BAC.*

**Theorem 15**

*The perpendicular bisectors of the sides of a triangle are concurrent and meet at a point, \( S \), called the circumcentre. (The circumcentre is the centre of the circle passing through the three vertices of the triangle.)*

**Given:** \( \triangle ABC \)

**To prove:**

The perpendicular bisectors of the sides are concurrent.

**Construction:**

Draw the perpendicular bisectors of \( AB \) and \( AC \) and let them meet at \( S \). It is now necessary to prove that the perpendicular bisector of \( BC \) also passes through \( S \).

**Proof:** Since \( S \) lies on the perpendicular bisector of \( AB \):
- \( SA = SB \)
Since \( S \) lies on the perpendicular bisector of \( AC \):
- \( SA = SC \)

\[ \therefore SB = SC \]

\[ \therefore S \text{ must lie on the perpendicular bisector of } BC \text{ and so the perpendicular bisectors of the sides of a triangle are concurrent. Since } SA = SB = SC, S \text{ is the centre of the circle passing through } A, B \text{ and } C. \]
Theorem 16

The bisectors of the angles of a triangle are concurrent and meet at a point, I, called the incentre. (The incentre is the centre of the circle which touches the three sides of the triangle.)

Given: \( \triangle ABC \)

To prove:

The bisectors of the angles are concurrent.

Construction:

Draw the bisectors of angles \( B \) and \( C \) and let them meet at \( I \). Draw \( IX, IY \) and \( IZ \) perpendicular to the sides \( AB, AC \) and \( BC \) respectively. It is now necessary to prove that the bisector of angle \( A \) also passes through \( I \).

Proof: Since \( I \) lies on the bisector of angle \( B \):

\[ IX = IZ \]

Since \( I \) lies on the bisector of angle \( C \):

\[ IY = IZ \]

\[ IX = IY \]

\[ \therefore \] I must lie on the bisector of angle \( A \) and so the bisectors of the angles of a triangle are concurrent. Since \( IX = IY = IZ \), \( I \) is the centre of the circle which touches the three sides of the triangle.

Midpoint theorem

The straight line joining the midpoints of two sides of a triangle is parallel to the third side and equal to \( \frac{1}{2} \) of its length.
**Theorem 17**
The medians of a triangle are concurrent and meet at a point, $G$, called the centroid. (A median is a straight line drawn from a vertex of a triangle to the midpoint of the opposite side.)

![Figure 15-58](image)

Given: $\Delta ABC$

To prove: The medians are concurrent.

Construction:

Draw the medians $BE$ and $CF$, and let them meet at $G$. Join $AG$ and produce it to meet $BC$ at $D$. It is now necessary to prove that $D$ is the midpoint of $BC$. Produce $AD$ to $H$, making $GH = AG$, and join $HB$ and $HC$.

Proof:

- In $\Delta AHC$, $EG \parallel CH$ (midpoint theorem)
- In $\Delta AHB$, $FG \parallel BH$ (midpoint theorem)

$\therefore$ $GBHC$ is a parallelogram.

$\therefore$ $BD = DC$ (diagonals of a parallelogram bisect each other)

$\therefore$ $AD$ is a median

**Theorem 18**
The altitudes of a triangle are concurrent and meet at a point, $H$, called the orthocentre.

![Figure 15-59](image)

Given: $\Delta ABC$

To prove: The altitudes are concurrent.

Construction:

Draw the altitudes, $BE$ and $CF$, and let them meet at $H$. Join $AH$ and produce it to meet $BC$ at $D$. Join $EF$. 
Proof: It is necessary to prove that $AD$ is perpendicular to $BC$.

$AEHF$ is a cyclic quadrilateral ($\angle AEH + \angle AFH = 180^\circ$).

$\therefore \angle FAH = \angle FEH$ (angles in the same segment on chord $FH$)

$BFEC$ is a cyclic quadrilateral ($\angle BFC = \angle BEC$)

$\therefore \angle FEB = \angle FCB$ (angles in the same segment on chord $FB$)

$\therefore \angle FAH = \angle HCD$

In $\triangle FAH$ and $\triangle CHD$:

$\angle FAH = \angle HCD$ (proved)

$\angle FHA = \angle DHC$ (vertically opposite)

$\therefore \angle AFH = \angle CDH$ (third angle of the triangles)

$\therefore \angle CDH = 90^\circ$ ($\angle AFH = 90^\circ$)

$\therefore AD$ is perpendicular to $BC$ and so the altitudes of the triangle are concurrent.

Exercises 15i

1 Prove each of the three concurrency theorems when $\triangle ABC$ is:
   a right-angled
   b obtuse-angled.

2 Show that, if $\triangle ABC$ is equilateral, the circumcentre, the incentre and the orthocentre coincide.

3 What can be said about the three points of concurrence, if $\triangle ABC$ is:
   a isosceles?
   b right-angled isosceles?

4 The altitudes $AD$, $BE$ and $CF$ of triangle $ABC$ meet at $H$. Prove that $H$ is the incentre of $\triangle DEF$.

5 $H$ is the orthocentre of $\triangle ABC$. If the altitude $AD$ is produced to meet the circumcircle at $Y$, prove that $HD = DY$.

6 $ABCD$ is a quadrilateral. What can be said about the quadrilateral if the bisectors of its angles are concurrent.

7 Study the proof of the concurrence of the medians of a triangle (Theorem 5) and show that the centroid divides each median in the ratio 2:1.