CHAPTER

11

Logic and reasoning

What problems are being solved?
From Alice's Adventures in Wonderland by Lewis Carroll.
Consider the following set of statements.
If I miss the train, I will be late for school.
I am late for school.
Therefore I missed the train.

These statements together form an argument. Arguments are central building blocks of mathematical proofs. To decide whether arguments are valid we must examine them in detail.

11.1 Propositions

The important components of arguments are propositions. A proposition is a sentence that can be held to be true or false.

Examples of propositions are:
- Melbourne is the capital of Victoria. (true)
- Tasmania is in the North of Australia. (false)
- \(2 + 2 = 4\) (true)
- \(2 + 3 = 6\) (false)

It is common practice to use lower case letters to represent propositions. So, we may use \(p\) to represent Melbourne is the capital of Victoria and \(q\) to represent \(2 + 2 = 4\). We shall often use the following notation to convey such meanings — the colon (:) stands for represents.

\[ p: \text{Melbourne is the capital of Victoria.} \]
\[ q: \text{2 + 2 = 4.} \]

Expressions such as ‘Stand over there’ or ‘Do you like roses?’ or ‘\(a + b\)’ are not propositions since they cannot be assigned true or false values.

A statement such as ‘The billionth digit in the decimal expansion of \(\pi\) is 7’ is a proposition whose truth or falsity is unknown.

It is customary to use T (true) and F (false) to symbolise the truth values of propositions.

\(m\): ‘the writer of this chapter is a male’, has the value T at all times.

\(r\): ‘it is raining’, has a value T or F depending on circumstances.

The important thing about propositions is that a truth value (T or F) can always be assigned, even though the values can be different on different occasions.

11.2 Negation, \(\neg p\)

A proposition may be negated by inserting the word not.

The negation of \(p\): ‘I am rich’ is symbolised by \(\neg p\) (not \(p\)) and means:

- I am not rich
- It is not true that I am rich
- The statement ‘I am rich’ is false
- or some other suitable expression.

So the truth values of a proposition \((p)\) and its negation \((\neg p)\) are opposites.
A convenient way of displaying such relationships is by means of a truth table such as Tables 11.1 and 11.2. Notice that both values, i.e. T and F, are allowed for in the table.

Table 11.1

<table>
<thead>
<tr>
<th>I like cats</th>
<th>I do not like cats</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 11.2

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \sim \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 11.2 uses the shorter form, using \( \rho \) to represent 'I like cats'.

A double negation is equivalent to the original proposition. For example, the proposition It is not true that I do not like cats may be symbolised as \( \sim ( \sim \rho ) \) and is equivalent to \( \rho \).

11.3 Set notation

We know that a set, \( P \), together with its complement, \( P' \) or \(~ P \), form a universal set defined by the particular context. Similarly a proposition \( p \) and its negation \( \sim p \) cover the whole possible situation. For this reason, Venn diagrams are sometimes used to display relationships between propositions.

Figure 11-1

\( P \) is the region associated with the proposition \( p \), and its complement \( P' \) or \(~ P \) is associated with the negation \( \sim p \). It is customary to use capital letters for set representations and lower case letters for the associated propositions.

Example 1

a Decide which of the following expressions are propositions.
b For each proposition write its negation, in words and in symbols.
c Represent each proposition, its negation, and its double negation in a truth table.

\( p: \) It is raining.
\( q: 5 + 4 \)
\( r: \) Can you swim?
\( s: \) Carlton is the best football team.

\( a \) \( p \) and \( s \) are propositions since they can be assigned truth values.

\( b \) \( \sim p: \) It is false that it is raining,
\( \sim s: \) It is false that Carlton is the best football team.

\( c \) Table 11.3

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \sim p )</th>
<th>( \sim ( \sim p ) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 11.4

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \sim s )</th>
<th>( \sim ( \sim s ) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

\( \sim ( \sim p ) \) and \( \sim ( \sim s ) \) are truth functions.
11.4 Conjunction, $p \land q$

Propositions joined by the word 'and' produce statements known as *conjunctions*.

Given $p$: The number is divisible by 3
$q$: The number is even

we write the conjunction of $p$ and $q$ as:

$p \land q$: The number is divisible by 3 and the number is even.

The conjunction is represented on the Venn diagram in Figure 11–2.

![Venn Diagram](image)

Figure 11–2

If $P$ represents the set of numbers divisible by 3 and $Q$ represents the even numbers, then the shaded intersection $P \cap Q$ represents those numbers which have both properties. Notice the similarity between the logical connection ($\land$) and the corresponding set operation ($\cap$).

Consider the propositions:

$p$: $5 + 8 = 2$
$q$: Pigs can't fly.
$r$: $a^2 - b^2 = (a - b)(a + b)$

The conjunction $p \land q$ states that '5 + 8 = 2 and pigs can't fly'. This conjunction is false.

If $p$ is true and $q$ is true, then $p \land q$ is true; otherwise $p \land q$ is false.

The conjunction $q \land r$ states that 'Pigs can't fly and $a^2 - b^2 = (a - b)(a + b)$'.

This conjunction is true because $q$ and $r$ are both true.

It can be seen that:

A conjunctive statement is only true when both the component propositions are separately true.

Notice the analogy with set intersection. The above information can be summarised in a truth table as shown in Table 11.5. The two columns on the left of the table contain all possible combinations of the truth values of $p$ and $q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
11.5 Disjunction, \( p \lor q \)

Two propositions joined by the word ‘or’ produce a disjunction.

Given \( p \): Jill is 17 years old
\( q \): It is raining

we write the disjunction of \( p \) and \( q \) as:

\( p \lor q \): Jill is 17 years old or it is raining, or both.

This disjunction is represented on the Venn diagram in Figure 11-3.

![Figure 11-3](image)

\( p \lor q \) is true if Jill is 17 years old, or if it is raining and also, naturally, if both are true.

The shaded union \( P \cup Q \) corresponds to the disjunctive statement \( p \lor q \). Notice the similarity between the symbols for disjunction (\( \lor \)) and union (\( \cup \)).

We have illustrated the inclusive use of the word or. Sometimes or is used to convey strict alternatives, e.g. ‘You will pay the fine or you will go to prison’. Here paying the fine is incompatible with going to prison — they cannot both apply. The symbol \( \lor \) is used to define ‘or’ in this exclusive sense.

A Venn diagram representation of \( p \lor q \) is shown in Figure 11-4. Notice that the area corresponding to \( P \cap Q \) is unshaded.

![Figure 11-4](image)

Consider the following set of propositions:

\( f \): The earth is flat.
\( t \): The Olympic Games were held in Melbourne in 1956.
\( r \): Australia is in the Southern hemisphere.
\( s \): \( 6 + 4 = 3 \)

Considering or in the inclusive sense, we see:

\( t \lor r \) is true, since both \( t \) and \( r \) are true
\( f \lor t \) is true, since \( f \) is false but \( t \) is true
\( r \lor f \) is true, since \( r \) is true although \( f \) is false
\( f \lor s \) is false, since \( f \) is false and \( s \) is false.
We notice that the only time a disjunctive statement $p \lor q$ is false is when $p$ is false and $q$ is false. This result is summarised in the truth table below (Table 11.6) in which $p$ and $q$ represent any two propositions.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

How is the truth table altered if the 'exclusive or' $(p \lor q)$ is intended? In the following work it will be assumed that or means the inclusive case unless otherwise stated.

**Example 2**
Which of the following sentences means the same as $x \in A \cup B$?

a $x \in A$ and $x \in B$

b $x \in A$ or $x \in B$ (exclusive)

c $x \in A$ or $x \in B$ (inclusive)

**Example 3**
Complete the following statements:

a $x \in A$ and $x \in B$ and $x \in C$ means the same as $x \in \ldots$

b $x \in A$ and either $x \in B$ or $x \in C$ means the same as $x \in \ldots$

We first draw a diagram with the three sets $A$, $B$ and $C$ overlapping as shown in Figure 11–6.
a Region 7 is the only one that belongs to all three sets as is required for \( x \in A \) and \( x \in B \) and \( x \in C \).

Since region 7 is the intersection of \( A, B \) and \( C \), the correct response is \( x \in A \cap B \cap C \).

b \( x \in A \) and either \( x \in B \) or \( x \in C \) means \( x \in A \) and \( x \in B \) or \( x \in A \) and \( x \in C \).

So \( x \in A \cap B \) or \( x \in A \cap C \) (replacing \( and \) with \( \cap \)).

So \( x \in (A \cap B) \cup (A \cap C) \) (replacing \( or \) with \( \cup \)).

This result may be inferred from the diagram by noting that:

\( A \cap B \) consists of region 7 + region 5

\( A \cap C \) consists of region 7 + region 4

and that the required area consists of region 7 + region 5 + region 4,

i.e. \( x \in A \cap (B \cup C) \).

**Example 4**

Let \( p \) be ‘Jill is 17’ and \( q \) be ‘Jill is Brian’s sister’. Write each of the following in symbolic form using \( p \) and \( q \).

a Jill is 17 and is Brian’s sister.

b Jill is 17 but is not Brian’s sister.

c Jill is neither 17 nor Brian’s sister.

d Jill is either not 17 or she is Brian’s sister.

e It is false that Jill is not 17 or not Brian’s sister.

f Jill is 17 or Jill is not 17 and is Brian’s sister.

\[
\begin{align*}
\text{a} & : p \land q \\
\text{b} & : p \land \neg q \\
\text{c} & : \neg p \land \neg q \\
\text{d} & : \neg p \lor q \\
\text{e} & : \neg (\neg p \lor \neg q) \\
\text{f} & : p \lor (\neg p \land q)
\end{align*}
\]

**Example 5**

Write a truth table for the compound statements:

a \( p \lor \neg q \)

b \( \neg (p \land \neg q) \)

\[
\begin{array}{cccc}
\hline
p & q & \neg q & p \lor \neg q \\
\hline
T & T & F & T \\
T & F & T & T \\
F & T & F & F \\
F & F & T & T \\
\hline
\end{array}
\]

Columns 1 and 2 give the four possible (T, F) combinations for \( p \) and \( q \).

Column 3 shows the negation of \( q \).

If we now consider Columns 1 and 3, and realise that \( p \lor \neg q \) is false only when \( p \) is false and \( \neg q \) is false, we can complete Column 4.

\[
\begin{array}{cccc}
\hline
p & q & \neg q & p \land \neg q & \neg (p \land \neg q) \\
\hline
T & T & F & F & F \\
T & F & T & T & F \\
F & T & F & F & T \\
F & F & T & F & F \\
\hline
\end{array}
\]

As in part a, complete Columns 1, 2 and 3. If we now consider that \( p \land \neg q \) is true only when both \( p \) and \( \neg q \) are true, we can complete Column 4. Then Column 5 is the negation of Column 4.
Exercises 11a

1 Which of the following statements are propositions?
   a \( 2 + 2 = 4 \)       b Jim is my brother
   c Where is my pen?       d The sky is blue
   e What a silly mistake!   f It is warm today
   g 20 is divisible by 6   h \( x = 3 \)

2 Let \( p \) be ‘John wears glasses’, and let \( q \) be ‘John is smart’. Write each of the following sentences in symbolic form.
   a John wears glasses and John is smart.
   b John does not wear glasses and is not smart.
   c John either wears glasses or is smart.
   d It is not true that John does not wear glasses.
   e Either John does not wear glasses or he is smart.
   f It is false that John wears glasses or is not smart.

3 Let \( p \) be ‘Joanne likes reading’ and let \( q \) be ‘Angelo likes reading’. Write verbal statements for each of the following:
   a \( \sim p \)
   b \( p \lor q \)
   c \( p \land q \)
   d \( \sim q \)
   e \( \sim p \lor \sim q \)
   f \( \sim (p \land q) \)
   g \( p \land \sim q \)
   h \( \sim (p \lor q) \)

4 Given \( p: \sqrt{25} = 5 \) (T) and \( q: \) Melbourne is in New South Wales (F), determine whether the following compound statements are true or false:
   a \( p \lor q \)
   b \( p \land q \)
   c \( \sim p \lor q \)
   d \( p \land \sim q \)
   e \( \sim p \lor \sim q \)
   f \( \sim (p \land q) \)
   g \( \sim p \land \sim q \)
   h \( p \land q \)

5 Let \( p \) be ‘It is hot’, and let \( q \) be ‘The sky is blue’.
   a Write verbal statements for each of the following:
      (i) \( p \lor \sim q \)
      (ii) \( p \land \sim q \)
      (iii) \( \sim (\sim p \lor q) \)
      (iv) \( \sim \sim p \)
   b If \( p \) is false and \( q \) is true, which of parts (i) to (iv) in Question 5a are true and which are false?

6 Complete the entries in each of the following truth tables:
   a
   \[
   \begin{array}{c|c|c|c|c|c}
   p & q & \sim p & \sim p \lor q & \sim q & \sim p \lor \sim q \\
   \hline
   T & T & F & T & F & T \\
   T & F & F & F & T & T \\
   F & T & T & T & F & F \\
   F & F & T & T & T & T \\
   \end{array}
   \]

   b
   \[
   \begin{array}{c|c|c|c|c|c|c}
   p & q & \sim p & \sim p \land q & \sim q & p \land \sim q \\
   \hline
   T & T & F & F & F & T \\
   T & F & F & F & F & T \\
   F & T & T & T & T & F \\
   F & F & T & T & T & T \\
   \end{array}
   \]
7 Construct a truth table for each of the following:
   a \( \sim (p \land q) \)
   b \( \sim (\sim p \land q) \)
   c \( (p \lor q) \lor \sim p \)
   d \( p \land \sim p \)
   e \( p \lor \sim p \)
   f \( (p \lor q) \lor p \)

8 Let \( p \) be ‘Scharlaine studies German’ and let \( q \) be ‘Scharlaine studies French’.
   a Write the following compound statement in symbolic form:
      ‘It is not true that Scharlaine studies German but not French’.
   b Construct a truth table for this compound statement.

9 a Construct a truth table for:
   (i) \( \sim (p \lor q) \)
   (ii) \( \sim p \land \sim q \)
   and show that they are the same.
   b Let \( p \) denote ‘I like tennis’, and \( q \) denote ‘I like music’, and so express (i) and (ii)
      in Question 9a as statements. Do they appear to be two ways of saying the same
      thing?

10 Construct a truth table for each of the following:
   a \( \sim (p \lor q) \lor \sim (q \lor p) \)
   b \( \sim (p \lor q) \land p \)
   c \( (p \lor q) \land (\sim p \lor \sim q) \)

11.6 Conditional statements, \( p \rightarrow q \)

When two propositions are joined by \( \textit{if . . . then . . .} \), the resulting statement is called the
\textit{conditional} or \textit{implication} of the two propositions. The symbol \( \rightarrow \) is used for implication.

Given
   \( p \): The ball has been bounced
   \( q \): The game has started
the implication \( p \rightarrow q \) is read as:
   \textit{If} the ball has been bounced \textit{then} the game has started.
The implication \( q \rightarrow p \) is read as follows:
   \textit{If} the game has started \textit{then} the ball has been bounced.
Consider the implication: \textit{If} I borrow the car \textit{then} I buy the petrol.
The component propositions may be written as:
   \( p \): I borrow the car
   \( q \): I buy the petrol.
The implication is represented as \( p \rightarrow q \). Suppose that \( p \) and \( q \) are true. Then \( p \rightarrow q \) is true
and we have:
   \( p \) (true), \( q \) (true); \( p \rightarrow q \) (true).
Suppose that I borrow the car but I do not pay for the petrol. In this case \( p \rightarrow q \) is false
and we have:
   \( p \) (true), \( q \) (false); \( p \rightarrow q \) (false).
Notice that the original statement tells me what happens if I do something. It does not say anything about what happens if I don’t do that particular thing. Suppose I don’t borrow the car but I do pay for the petrol. This does not contradict the original statement so we have:

\[ p \text{ (false), } q \text{ (true)}; \quad p \rightarrow q \text{ (true)}. \]

Also, if I don’t borrow the car and I don’t buy the petrol, my original statement is not contradicted. So we have:

\[ p \text{ (false), } q \text{ (false)}; \quad p \rightarrow q \text{ (true)}. \]

These results may be summarised in a truth table as follows.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The only time an implication \( p \rightarrow q \) is false is when \( p \) is true and \( q \) is false.

Figure 11–7 shows how to represent an implication on a Venn diagram. The shaded region shows where \( p \rightarrow q \) is true.

1 represents \( p \) is true and \( q \) is true.
2 represents \( p \) is true and \( q \) is false.
3 represents \( p \) is false and \( q \) is true.
4 represents \( p \) is false and \( q \) is false.

11.7 Converse, inverse and contrapositive

Given the conditional statement \( p \rightarrow q \), the **converse** of the statement is \( q \rightarrow p \), the **inverse** of the statement is \( \sim p \rightarrow \sim q \), the **contrapositive** of the statement is \( \sim q \rightarrow \sim p \).

Suppose:

\[ p: \text{ The number ends in zero.} \]
\[ q: \text{ The number is divisible by five.} \]
\[ p \rightarrow q: \text{ If the number ends in zero, then the number is divisible by five.} \]
\[ q \rightarrow p: \text{ If the number is divisible by five, then the number ends in zero.} \]
\[ \sim p \rightarrow \sim q: \text{ If the number does not end in zero, then the number is not divisible by five.} \]
\[ \sim q \rightarrow \sim p: \text{ If the number is not divisible by five, then the number does not end in zero.} \]
We note that the original statement \((p \rightarrow q)\) happens to be true. We observe that the contrapositive is also true, but that the converse and inverse are false. Draw Venn diagrams to depict the converse, inverse, and contrapositive statements. Show that the contrapositive diagram is identical in form to that of Figure 11–7.
These relationships will be explored more fully in a later section.

**11.8 Equivalence, \(p \leftrightarrow q\)**

Propositions \(p\) and \(q\) are equivalent \((p \leftrightarrow q)\) when they are both true together and both false together.
For example, the following propositions are equivalent:

\[
p: \text{A number is even} \quad q: \text{A number is divisible by 2}
\]

\(p \leftrightarrow q\) is read \(p\ if\ and\ only\ if\ q\).
In this example we would write:

\(A\ number\ is\ even\ if\ and\ only\ if\ it\ is\ divisible\ by\ 2.\)

Table 11.8 and Figure 11–8 respectively show the truth table and a Venn diagram to represent equivalence.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(p \leftrightarrow q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**Example 6**

Given the propositions:

\(p: \text{This subject is difficult} \quad q: \text{I don’t study hard}\)

\(p \rightarrow q: \text{If this subject is difficult, then I don’t study hard.}\)
\(q \rightarrow p: \text{If I don’t study hard, then this subject is difficult.}\)
\(p \leftrightarrow q: \text{This subject is difficult if and only if I don’t study hard.}\)
\(\sim q \rightarrow \sim p: \text{If I do study hard, then this subject is not difficult.}\)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(p \rightarrow q)</th>
<th>(q \rightarrow p)</th>
<th>(p \leftrightarrow q)</th>
<th>(\sim q \rightarrow \sim p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
<td>T</td>
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<td>T</td>
</tr>
</tbody>
</table>
Notice that the contrapositive $\neg q \rightarrow \neg p$, being an implication, is false only when $\neg q$ is true and $\neg p$ is false, i.e. $q$ is false and $p$ is true. This demonstrates its identity with $p \rightarrow q$.

The shaded region representing the truth of $p \leftrightarrow q$ is the area of shading common to $p \rightarrow q$ and $q \rightarrow p$.

Notes on implication:

1. Within mathematics there is special interest in implications that are true for all objects under consideration.

   If $p \rightarrow q$ is true and if $p$ is true then, as indicated in the truth table, the truth of $q$ is guaranteed. This condition 'if $p$ then $q$' is also written as 'p is sufficient for $q$'.

   Reference to the truth table further indicates that, when $p \rightarrow q$ is true, we cannot have $p$ true unless $q$ is true. This condition 'p only if $q$' is also written as 'q is necessary for $p$'.

   When $p \rightarrow q$ and $q \rightarrow p$ (i.e. $p \leftrightarrow q$) we write ‘$p$ if and only if $q$’ or alternatively ‘$p$ is necessary and sufficient for $q$’.

2. In mathematical contexts the notation $\Rightarrow$ and $\Leftrightarrow$ is commonly used instead of $\rightarrow$ and $\leftrightarrow$. The following illustrate familiar uses of the symbols:

   $$x = 3 \Leftrightarrow x + 8 = 11$$
   $$x = 3 \Rightarrow x^2 = 9$$

   Clearly $x = 3$ is sufficient for $x^2 = 9$, while $x^2 = 9$ is necessary for $x = 3$. Notice that $x^2 = 9$ is not sufficient for $x = 3$. Why? Why is it appropriate to use $\leftrightarrow$ in the other statement?

3. If $P$ and $Q$ respectively denote the truth sets of $p$ and $q$ (i.e. sets whose members have properties $p$ and $q$), the preceding relationships may be represented as in Figure 11-10. Notice that we are here concerned with true statements so the diagram differs from Figure 11-7, illustrating $p \rightarrow q$, where allowances are made for all combinations of true and false.

   $P \subseteq Q$ which illustrates that if $x \in P$ then $x \in Q$, i.e. $p$ is sufficient for $q$. Also $x$ cannot belong to $P$ unless $x \in Q$ so that $q$ is necessary for $p$.

   Draw a diagram to illustrate necessary and sufficient conditions.

4. Our definitions lead us to wider applications than we are accustomed to within mathematics, where the context is usually restricted by the nature of the problem.
Since we allow \( \rightarrow \) and \( \leftrightarrow \) to be defined by truth tables we agree to accept implications that seem strange by normal standards. Since \( p \rightarrow q \) is false only when \( p \) is true and \( q \) is false, we accept, for example:

- 'If babies are small humans then Australia is a continent'
- 'If puppies are small humans then Australia is a continent'

as true implications.

On the other hand:

- 'If babies are small humans then Australia is in the northern hemisphere'

is false.

**Summary:**

<table>
<thead>
<tr>
<th>Term</th>
<th>Connective</th>
<th>Translation</th>
<th>Truth table</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negation</td>
<td>( \sim )</td>
<td>not</td>
<td>Table 11.2</td>
</tr>
<tr>
<td>Conjunction</td>
<td>( \wedge )</td>
<td>and</td>
<td>Table 11.5</td>
</tr>
<tr>
<td>Disjunction</td>
<td>( \vee )</td>
<td>or</td>
<td>Table 11.6</td>
</tr>
<tr>
<td>Implication</td>
<td>( \rightarrow )</td>
<td>if . . . then . . .</td>
<td>Table 11.7</td>
</tr>
<tr>
<td>Equivalence</td>
<td>( \leftrightarrow )</td>
<td>. . . if and only if . . .</td>
<td>Table 11.8</td>
</tr>
</tbody>
</table>

It is recommended that you commit to memory the truth tables for the connectives above.

It is important to feel comfortable translating between verbal and symbolic statements, e.g. between truth tables, Venn diagrams and verbal statements.

**Example 7**

Let \( p \) be the sentence 'xis prime', let \( q \) be the sentence 'xis odd', and let \( r \) be the sentence 'xis prime and x is odd'.

a Translate the sentence \( (p \land q) \rightarrow r \) into words.

b Translate the sentence 'If x is a prime number greater than 2, then x is odd' into symbolic form.

**A**

a \( p \land q \) means that 'xis prime and x is odd'. So \( p \land q \rightarrow r \) means 'If x is prime and x is odd then x > 2'.

b 'xis a prime number greater than 2' means x is prime and \( x > 2 \), i.e. \( p \land r \). So the given implication is \( p \land r \rightarrow q \).

**Example 8**

Let \( p \) be 'The weather is hot' and \( q \) be 'The wind blows' and \( r \) be 'It is summertime'. Write the following statements in symbolic form.

a If the wind blows, then the weather is hot.

b The wind blows if and only if it is summertime and the weather is hot.

c If the wind does not blow, then it is not summertime.

d It is summertime if and only if the weather is hot.

**A**

a \( q \rightarrow p \)

b \( q \leftrightarrow (r \land p) \)

c \( \sim q \rightarrow \sim r \)

d \( r \leftrightarrow p \)
Example 9
Construct truth tables for \( \neg (p \land q) \) and \( \neg p \lor \neg q \) and use them to show that:
\[
\neg (p \land q) \leftrightarrow \neg p \lor \neg q
\]

Table 11.10

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( p \land q )</th>
<th>( \neg (p \land q) )</th>
<th>( \neg p \lor \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>( \neg ) T</td>
<td>( \neg ) F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>( \neg ) T</td>
<td>( \neg ) T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>( \neg ) T</td>
<td>( \neg ) T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>( \neg ) T</td>
<td>( \neg ) T</td>
</tr>
</tbody>
</table>

Beginning with all possible (T, F) combinations for \( p \) and \( q \) in the first two columns, we then compile the entries for \( \neg p \), \( \neg q \) and \( p \land q \) using the results previously established. Entries for \( \neg (p \land q) \) are obtained by negating entries for \( p \land q \). Entries for \( \neg p \lor \neg q \) are obtained by combining the entries in Columns 3 and 4, using the disjunction pattern. Since the last two columns are identical, we see that \( \neg (p \land q) \) and \( \neg p \lor \neg q \) are equivalent statements, i.e. \( \neg (p \land q) \leftrightarrow \neg p \lor \neg q \).

Common sense confirms that this result is to be expected. For example, suppose:

\( p \): I have brown eyes
\( q \): I drive a car

\( p \land q \) means that I have brown eyes and I drive a car.

If I do not have brown eyes (\( \neg p \)) or if I do not drive a car (\( \neg q \)), the conjunction is negated. This is precisely the meaning of \( \neg p \lor \neg q \).

Example 10
Construct a truth table to show that \( \neg (p \lor q) \leftrightarrow \neg p \land \neg q \).

Table 11.11

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \lor q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( \neg (p \lor q) )</th>
<th>( \neg p \land \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>( \neg ) F</td>
<td>( \neg ) F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>( \neg ) F</td>
<td>( \neg ) T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>( \neg ) F</td>
<td>( \neg ) T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>( \neg ) F</td>
<td>( \neg ) T</td>
</tr>
</tbody>
</table>

Proceeding as before we assign all possible combinations of T and F values to \( p \) and \( q \) in Columns 1 and 2. We then complete Columns 3, 4, and 5, using the pattern for disjunction and negation. Entries for \( \neg (p \lor q) \) are obtained by negating the entries in Column 3. Entries for \( \neg p \land \neg q \) are obtained by applying the conjunction pattern to the entries in Columns 4 and 5.

Since Columns 6 and 7 contain identical entries we infer that:
\[
\neg (p \lor q) \leftrightarrow \neg p \land \neg q.
\]

Make up an everyday example that confirms the sense of this result and verify that the shaded region of the following Venn diagram illustrates the equivalence.
Example 11
Construct a truth table to show that \( p \to q \) and \( \neg p \lor q \) are equivalent statements.

\[
\begin{array}{|c|c|c|c|c|}
\hline
p & q & \neg p & \neg p \lor q & p \to q \\
\hline
T & T & F & T & T \\
T & F & F & F & F \\
F & T & T & T & T \\
F & F & T & T & T \\
\hline
\end{array}
\]

Column 5 contains the standard implication pattern for \( p \to q \). Column 4 is found by means of the disjunction pattern applied to \( \neg p \) and \( q \). Since Columns 4 and 5 contain identical entries, we see that \( p \to q \) and \( \neg p \lor q \) are equivalent statements.

Note:
Equivalent statements mean that they are interchangeable. Illustrate the equivalence by means of a Venn diagram in which the region \( \neg P \cup Q \) is shaded.

Example 12
Let \( p \) be ‘John is smart’, let \( q \) be ‘John studies Mathematics’ and let \( r \) be ‘John studies Physics’.

Construct a truth table for the compound statement:
‘If John is smart, then he studies either Mathematics or Physics’.

We are asked to construct a truth table for:
\[ p \to (q \lor r) \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
p & q & r & q \lor r & p \to (q \lor r) \\
\hline
T & T & T & T & T \\
T & T & F & T & T \\
T & F & T & T & T \\
T & F & F & F & F \\
F & T & T & T & T \\
F & T & F & T & T \\
F & F & T & T & T \\
F & F & F & F & T \\
\hline
\end{array}
\]

Note that, since we have three symbols \( p, q \) and \( r \), our truth table starts with three columns containing eight different combinations of T and F.
Column 4 is derived from Columns 2 and 3, using the pattern for disjunction.
Column 5 is derived from Columns 1 and 4 using the pattern for implication.
Exercises 11b

1 Given  
   \[ p: \text{The positive square root of 9 is 3} \]
   \[ q: \text{Parallel lines never meet.} \]
   a write statements for each of the following:
      (i) \( p \implies q \)
      (ii) \( p \iff q \)
      (iii) \( \neg q \implies \neg p \)
   b construct truth tables for the cases listed in Question 1a.
   c draw Venn diagrams to illustrate the cases listed in Question 1a.

2 Classify these sentences as simple propositions (P), conjunctions (C), disjunctions (D or D), implications (I), or equivalences (E). Symbolise the sentences using the underlined letters as symbols for the relevant propositions.
   a Fish swim.
   b Robert Menzies was a politician or England is a republic.
   c \( \sqrt{4} = 2 \text{ and } \log_8 1 = 0. \)
   d Cars are stolen if left uninked.
   e Animals breathe and plants do not have feelings.
   f If you smoke heavily, then you die at a younger age.
   g A triangle is equilateral if and only if it has equal angles.
   h The grass is wet.
   i You will catch a cold if you sleep in wet clothes.
   j Either you will sleep on your side or I will wake you up.

3 For the statements in Question 2 that are implications, write, in words, their converses, inverses and contrapositives.

4 Let \( p \) be the sentence ‘\( x^2 = 9 \)’ and let \( q \) be the sentence ‘\( y \leq 1 \)’ and let \( r \) be the sentence ‘\( \pi \) is irrational’.
   Translate into words:
      a \((p \lor q) \implies r\)
      b \((p \land q) \implies r\)
      c \((r \land p) \iff q\)
      d \(r \implies \neg(p \lor q)\)
      e \(\neg r \land p \implies q\)

5 Translate into symbols:
   a ‘If \( x^2 = 9 \) or \( y \leq 1 \), but not both, then \( \pi \) is irrational.’
   b ‘If \( \pi \) is not irrational then \( x^2 = 9 \) and \( y \leq 1 \).’
   c ‘If \( \pi \) is rational or \( x^2 = 9 \) then \( y > 1 \).’
   d ‘\( \pi \) is irrational if and only if \( y > 1 \) or \( x^2 = 9 \), but not both.’

6 a Copy and complete the following truth table.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \implies q )</th>
<th>( q \implies p )</th>
<th>( \neg p \implies \neg q )</th>
<th>( \neg q \implies \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
b What can we conclude?
c Consider the statements:
   \( p \): Triangle \( ABC \) is equilateral
   \( q \): Triangle \( ABC \) is isosceles.

Which of the following are true:
(i) \( p \rightarrow q \)
(ii) \( q \rightarrow p \)
(iii) \( \neg p \rightarrow \neg q \)
(iv) \( \neg q \rightarrow \neg p \)

7 Let \( p \) denote ‘It is springtime’ and \( q \) denote ‘The sky is blue’. Write the following statements in symbolic form:
a The sky is blue only if it is springtime.
b The sky is not blue when it is not springtime.
c Whenever it is springtime, the sky is blue.
d A necessary condition for the sky to be blue, is that it be springtime.

8 Write truth tables for each of the following:
   a \( (p \land q) \rightarrow p \)
   b \( (p \rightarrow q) \land (q \rightarrow p) \)
   c \( (p \rightarrow q) \rightarrow (p \land q) \)
   d \( (p \lor r) \land (p \rightarrow q) \)

9 Write truth tables for:
   a \( p \leftrightarrow q \)
   b \( (p \rightarrow q) \land (q \rightarrow p) \)

What do you conclude?

10 State whether the following statements are true or false:
   a If \( 3 + 3 = 6 \), then \( 5 + 5 = 10 \)
   b If \( 3 + 3 = 6 \), then \( 5 + 5 = 12 \)
   c If \( 3 + 3 = 8 \), then \( 5 + 5 = 10 \)
   d If \( 3 + 3 = 8 \), then \( 5 + 5 = 12 \)

11 State whether the following statements are true or false:
   a \( 3 + 3 = 6 \) if and only if \( 5 + 5 = 10 \)
   b \( 3 + 3 = 6 \) if and only if \( 5 + 5 = 12 \)
   c \( 3 + 3 = 8 \) if and only if \( 5 + 5 = 10 \)
   d \( 3 + 3 = 8 \) if and only if \( 5 + 5 = 12 \)

12 Let \( p \) be ‘Magda is happy’, \( q \) be ‘Lisa is sad’ and \( r \) be ‘Greg works hard’.
   Write each of the following in symbolic form and construct a truth table in each case.
   a Greg works hard if and only if Magda is happy and Lisa is not sad.
   b If Greg does not work hard then Magda is not happy and Lisa is sad.

13 A truth table with two simple statements has four rows, and one with three statements has eight rows. How many rows would there be if there were:
   a four simple statements?
   b five simple statements?
   c \( n \) simple statements?
14 Write, in words and in symbols, logical statements that are represented by the following diagrams:

\[ Q \implies P \]

\[ A \implies B \]

\[ S \implies T \]

\[ L \implies M \]

\[ A \implies B \]

\[ W \implies V \]

11.9 Tautologies

Sometimes we use expressions such as ‘It will be fine today or else it will rain’. We have just said something that will be true no matter what the day is like. Such statements are called tautologies. We shall be concerned with logical tautologies.

A tautology is a statement that is always true regardless of the truth of the component statements.

\( p \lor q, p \land q \) and \( p \to q \) are not tautologies since there is at least one false (F) entry in the truth table column. However \( p \lor \neg p \) is a tautology. The truth table is shown below (Table 11.14).

**Table 11.14**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
<th>( p \lor \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Clearly \( p \lor \neg p \) is always true irrespective of the truth value of \( p \). For example, if \( p \) represents the statement ‘this number is prime’ then \( p \lor \neg p \) means ‘either this number is prime or it is not prime’.

This is an obviously true statement whether or not \( p \) is true. So we infer that:

A statement which has only T in each row of the last column of its truth table is a tautology.
Example 13

Show that:

a. $p \rightarrow (p \lor q)$ is a tautology

b. $(p \rightarrow q) \rightarrow (\neg p \lor q)$ is a tautology

c. $(p \land q) \lor \neg p$ is not a tautology.

A

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
<th>$p \rightarrow (p \lor q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The entries in Column 4 are obtained by applying the implication pattern to the entries in Columns 1 and 3. Since Column 4 contains all T values, it follows that $p \rightarrow (p \lor q)$ is a tautology.

b

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$p \rightarrow q$</th>
<th>$\neg p \lor q$</th>
<th>$(p \rightarrow q) \rightarrow (\neg p \lor q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The entries in Columns 3, 4 and 5 are obtained from the entries in Columns 1 and 2, using the standard patterns for $\neg$, $\rightarrow$ and $\lor$. The entries in Column 6 are obtained by applying the implication pattern to the entries in Columns 4 and 5. Since Column 6 contains only ‘T’ s it follows that $(p \rightarrow q) \rightarrow (\neg p \lor q)$ is a tautology.

c

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$p \land q$</th>
<th>$(p \land q) \lor \neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The entries in Columns 3 and 4 are obtained from the entries in Columns 1 and 2, using the standard patterns for $\neg$ and $\land$. The entries in Column 5 are obtained by combining the entries in Columns 3 and 4 according to the disjunction pattern. We see that $(p \land q) \lor \neg p$ is false when $p$ is true and $q$ is false, so that it is not a tautology.

We may illustrate the meaning of a, b and c above through an example which enables the symbols to be translated into verbal expressions.

Suppose $p$ stands for the sentence ‘Roses are flowers’, and that $q$ stands for the sentence ‘$1 = 2$’.

The statement $p \rightarrow (p \lor q)$ means ‘Roses are flowers implies that roses are flowers or $1 = 2$.’
Alternatively this may be written as ‘If roses are flowers, then roses are flowers or \(1 = 2\)’.
The statement \((p \rightarrow q) \rightarrow (\neg p \lor q)\) can be written in several ways, e.g. ‘If (roses are flowers implies \(1 = 2\)) then (roses are not flowers or \(1 = 2\))’.
The statement \((p \land q) \lor \neg p\) means ‘Roses are flowers and \(1 = 2\) or roses are not flowers’.

**Example 14**
Show that \((p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r)\) is a tautology.

Here we have three symbols, so our truth table commences with three columns containing the eight different combinations of T and F values for \(p\), \(q\) and \(r\). These are set out systematically beginning with TTT.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(p \rightarrow q)</th>
<th>(q \rightarrow r)</th>
<th>((p \rightarrow q) \land (q \rightarrow r))</th>
<th>((p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Columns 4, 5 and 7 are filled out using the implication patterns.

Column 6 is filled by application of the conjunction pattern to the entries in Columns 4 and 5.

Column 8 is filled out by applying the implication pattern to the entries in Columns 6 and 7. Since no (T, F) combination occurs, the entries in Column 8 are all ‘T’s.

So the statement is a tautology.

This tautology is often used in mathematical situations. It says that \((p\) implies \(q\)) and \((q\) implies \(r\)) implies \((p\) implies \(r\)), i.e. If \((p\) implies \(q\)) and \((q\) implies \(r\)) then \((p\) implies \(r\)).
For example, suppose \(p\) represents the sentence ‘\(2x + 1 = 5\)’, \(q\) represents the sentence ‘\(2x = 4\)’, and \(r\) represents the sentence ‘\(x = 2\)’.

Translating \(\rightarrow\) as ‘implies’, the statement would read:
\[2x + 1 = 5 \text{ implies } 2x = 4\]
and
\[2x = 4 \text{ implies } x = 2\]
implies \[2x + 1 = 5 \text{ implies } x = 2\]

If \(r\) represents the sentence ‘\(x = 5\)’ then the statement reads:
\[2x + 1 = 5 \text{ implies } 2x = 4\]
and
\[2x = 4 \text{ implies } x = 5\]
implies \[2x + 1 = 5 \text{ implies } x = 5\]

Do not confuse an incorrect mathematical answer with invalid reasoning. \(x = 5\) is not the correct solution to \(2x + 1 = 5\). However, the conditional statement remains correct as a piece of logic. Read the statement again carefully to be certain on this point.
Exercises 11c

1 Using truth tables determine which of the following statements are tautologies.

a. \( p \rightarrow \sim q \)

b. \( \sim (q \land \sim q) \)

c. \( p \land q \rightarrow p \)

d. \( p \leftrightarrow \sim \sim p \)

e. \( [(p \rightarrow q) \land p] \rightarrow q \)

f. \( (p \lor q) \rightarrow (p \land q) \)

g. \( (q \rightarrow p) \leftrightarrow (p \rightarrow q) \)

h. \( [(p \rightarrow q) \land \sim q] \rightarrow \sim p \)

i. \( \sim (p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \land (q \rightarrow p)] \)

j. \( (p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p) \)

k. \( [(p \rightarrow q) \land \sim q] \rightarrow \sim p \)

l. \( [p \land (p \rightarrow q)] \rightarrow q \)

m. \( [p \rightarrow (q \land \sim q)] \leftrightarrow \sim p \)

n. \( [p \land (q \lor r)] \leftrightarrow ([p \land q] \lor (p \land r)] \)

o. \( [p \rightarrow (q \rightarrow r)] \leftrightarrow [(p \rightarrow q) \rightarrow r] \)

11.10 Negation of compound sentences

We shall first construct a truth table for \( \sim (p \land q) \), i.e. the negation of \( p \land q \); together with a truth table for \( \sim p \lor \sim q \) (Table 11.19).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( p \land q )</th>
<th>( \sim (p \land q) )</th>
<th>( \sim p \lor \sim q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>T</td>
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<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Since the last two columns are identical we see that \( \sim (p \land q) \) and \( \sim p \lor \sim q \) are equivalent statements, i.e. \( \sim (p \land q) \leftrightarrow \sim p \lor \sim q \) is a tautology.

The negation of a conjunction \( \sim (p \land q) \) is the same as the disjunction of the negations \( \sim p \lor \sim q \).

Common sense confirms that this result is to be expected. For example, suppose:

- \( p \): I have brown eyes
- \( q \): I drive a car

\( p \land q \) means that I have brown eyes and I drive a car.

If I do not have brown eyes \( (\sim p) \) or if I do not drive a car \( (\sim q) \), the conjunction is negated. This is precisely the meaning of \( \sim p \lor \sim q \).
Example 15
Construct a truth table to show that \(-(p \lor q) \leftrightarrow -p \land -q\).

Table 11.20

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(p \lor q)</th>
<th>(-p)</th>
<th>(-q)</th>
<th>(-(p \lor q))</th>
<th>(-p \land -q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Since Columns 6 and 7 contain identical entries we infer that:
\(-(p \lor q) \leftrightarrow -p \land -q\) is a tautology.

The negation of a disjunction \(\sim(p \lor q)\) is the same as the conjunction of the negations \(\sim p \land \sim q\).

Similar results can be established for an implication and an equivalence. The negations are shown in the following table.

Table 11.21

<table>
<thead>
<tr>
<th>Term</th>
<th>Formula</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjunction</td>
<td>(p \land q)</td>
<td>(-p \lor -q)</td>
</tr>
<tr>
<td>Disjunction</td>
<td>(p \lor q)</td>
<td>(-p \land -q)</td>
</tr>
<tr>
<td>Implication</td>
<td>(p \rightarrow q)</td>
<td>(p \land \sim q)</td>
</tr>
<tr>
<td>Equivalence</td>
<td>(p \leftrightarrow q)</td>
<td>(\sim p \leftrightarrow q \lor p \leftrightarrow -q)</td>
</tr>
</tbody>
</table>

Use truth tables to illustrate the negations for implication and equivalence.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>It is summertime and the weather is hot.</td>
<td>It is not summertime or the weather is not hot.</td>
</tr>
<tr>
<td>Jill is 17 or Tom is 18.</td>
<td>Jill is not 17 and Tom is not 18.</td>
</tr>
<tr>
<td>If I am helping other people, then I am happy.</td>
<td>I am helping other people and I am not happy.</td>
</tr>
<tr>
<td>A number is even if and only if it is divisible by 2.</td>
<td>A number is not even if and only if it is divisible by 2 or a number is even if and only if it is not divisible by 2.</td>
</tr>
</tbody>
</table>
Exercises 11d

1. With the aid of truth tables show that:
   a. $p \land \neg q$ is the negation of $p \rightarrow q$.
   b. $\neg p \leftrightarrow q$ or $p \leftrightarrow \neg q$ is the negation of $p \leftrightarrow q$.

2. Write the negation of each of the following:
   a. $p \lor \neg q$
   b. $\neg p \land q$
   c. $\neg p \rightarrow q$
   d. $\neg p \leftrightarrow \neg q$
   e. $(p \land q) \lor r$
   f. $p \lor (q \lor r)$
   g. $(p \lor q) \rightarrow r$
   h. $(p \land \neg q) \rightarrow r$
   i. $p \land (q \rightarrow r)$

3. Write the negation of each of the following:
   a. $x = 4$ and $y = 2$
   b. $x = 4$ if $y \neq 2$
   c. $x = 4$ or $y \neq 2$
   d. $x \neq 4$ or $y \neq 2$
   e. $x = 4$ is equivalent to $y = 2$

4. Write the negation of each of the following statements.
   a. My hair is dark and my eyes are not blue.
   b. If two triangles are congruent, then they are similar.
   c. If $2x + 1 = 5$, then $x = 2$.
   d. Pigs can’t fly and $a^2 - b^2 = (a - b)(a + b)$.
   e. If Ben smokes or drinks, he will not be fit.
   f. The sum of the angles of a polygon is $180^\circ$ if and only if the polygon is a triangle.
   g. When it is hot, I open the window and turn on the fan.
   h. Jill is either not 17 or she is Brian’s sister.
   i. Jill is 17 or Jill is not 17 and is Brian’s sister.
   j. The wind blows if and only if it is summertime and the weather is hot.
   k. If John is smart, then he studies either Mathematics or Physics.

11.11 Validity of arguments

We are now ready to evaluate the validity of arguments such as the one presented at the beginning of this chapter.

An argument is a series of assertions called premises which are always assumed true, followed by a final statement called the conclusion. The premises and the conclusion are propositions.

An argument is valid if we can force the conclusion to take only a true value. If the conclusion may be false, then the argument is invalid.

Note that the terms true and false are associated with propositions. The terms valid and invalid are associated with arguments.

An argument is said to be valid if and only if the conjunction of the premises implies the conclusion, i.e. if the premises are all true, the conclusion must also be true.

Example 16

Investigate the validity of the following arguments using the underlined letters to symbolise the statements.

a. If I miss the train, I will be late.
   I missed the train.
   Therefore I will be late.

b. If I miss the train, I will be late.
   I am late.
   Therefore I missed the train.
The premises of the argument are \( t \to I \) and \( I \).

The conclusion of the argument is \( I \).

The complete truth table for the implication \( t \to I \) is shown in Table 11.22.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( I )</th>
<th>( t \to I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Now, since the premises of an argument are assumed to be true, we are only concerned with Row 1 of the truth table. (This is the only row for which both premises, i.e. \( t \) and \( t \to I \), have the value T.) The pattern in this row shows that the entry for \( I \) is forced to be true when the entries for \( t \) and \( t \to I \) are true. Since \( I \) is the conclusion, we infer that the argument is valid.

Sometimes the detail of the argument is set out as below, in which the premises are written above the bar and the conclusion below it.

\[
\begin{align*}
& \quad t \\
& \quad t \to I \\
& \therefore \quad I
\end{align*}
\]

b The argument in this case may be set out as below.

\[
\begin{align*}
& \quad I \\
& \quad I \\
& \therefore \quad I
\end{align*}
\]

By reference to the truth table, Table 11.22, we see that both of the rows 1 and 3 are compatible with the premises. The conclusion, \( I \), is not forced to be true (it can be false as in Row 3), and so the argument is invalid.

It is useful to verify the sense of the solution by considering the real situation. For example, the train might break down so that I will still be late even though I did not miss the train.

**Example 17**

Determine whether the following arguments are valid:

**a** \( p \leftrightarrow q \)

\[
\begin{align*}
& \quad p \\
& \quad \neg p \\
& \therefore \quad q \quad \therefore \quad \neg q
\end{align*}
\]

**b** \( p \to q \)

\[
\begin{align*}
& \quad p \to q \\
& \quad \neg p \\
& \therefore \quad q
\end{align*}
\]

a Consider the truth table of the argument.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \to q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Both premises are true only in the first row, in which the conclusion is also true and so the argument is valid. For example:

\[ p \iff q: \text{The base angles of a triangle are equal if and only if the triangle is isosceles.} \]
\[ p: \text{The base angles are equal.} \]
\[ \therefore q: \text{The triangle is isosceles.} \text{ (Valid)} \]

**Example 18**
Determine whether the following argument is valid:

If I miss the train, I will be late for school.
If I am late for school, the teacher will not be happy.
Therefore, if I miss the train, the teacher will not be happy.

The argument can be set out as follows:

\[ p \rightarrow q \]
\[ q \rightarrow r \]
\[ \therefore p \rightarrow r \]

Consider the truth table for the argument.

**Example 18**
Determine whether the following argument is valid:

If I miss the train, I will be late for school.
If I am late for school, the teacher will not be happy.
Therefore, if I miss the train, the teacher will not be happy.

The argument can be set out as follows:

\[ p \rightarrow q \]
\[ q \rightarrow r \]
\[ \therefore p \rightarrow r \]

Consider the truth table for the argument.
Example 19
Determine whether the following argument is valid.

\[ \begin{align*}
\neg p \land \neg q \\
r \rightarrow p \\
\therefore \quad r
\end{align*} \]

The premises are \( \neg p \land \neg q \), and \( r \rightarrow p \). The conclusion is \( r \).

Tables 11.26 and 11.27 contain truth tables for the premises.

Since the premises are true, we need include only those parts of the truth tables that meet this requirement.

<table>
<thead>
<tr>
<th>Table 11.26</th>
<th>Table 11.27</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Note:
Students should construct the complete truth tables and verify that the entries shown are the only rows for which all premises are true.

Since \( \neg p \land \neg q \) is true we know that \( \neg p \) and \( \neg q \) are true. From the negation property, we now infer that \( p \) and \( q \) are false.

We now have two known entries for the truth table in Table 11.27 and the form of the table indicates that \( r \) must be false.

Since \( r \) is the conclusion, the argument is invalid.

Write verbal statements for \( p \), \( q \) and \( r \) to illustrate the meaning of the invalid argument in a particular case.

11.12 Use of tautologies

In Examples 16 to 19, we used the fact that an argument is valid if the conjunction of the premises implies the conclusion. This means then that, if the implication is a tautology, then the argument is valid. We will illustrate this for Example 17.

Example 20
With the use of tautologies, determine whether the following arguments are valid (See Example 15).

a \( p \leftrightarrow q \)

\[ \begin{align*} 
p \\
\therefore \quad q
\end{align*} \]

b \( p \rightarrow q \)

\[ \begin{align*} 
\neg p \\
\therefore \quad \neg q
\end{align*} \]

\( (p \leftrightarrow q) \land p \rightarrow q \)

Construct a truth table for:

<table>
<thead>
<tr>
<th>Table 11.28</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>F</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>
Since \[(p \leftrightarrow q) \land p \rightarrow q\] is a tautology, the argument is valid.

b Construct a truth table for:

\[(p \rightarrow q) \land \neg p \rightarrow \neg q\]

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(\neg p)</th>
<th>(\neg q)</th>
<th>(p \rightarrow q)</th>
<th>((p \rightarrow q) \land \neg p)</th>
<th>((p \rightarrow q) \land \neg p \rightarrow \neg q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Since \[(p \rightarrow q) \land \neg p \rightarrow \neg q\] is not a tautology, the argument is not valid.

**Exercises 11e**

1. Determine whether the following arguments are valid:
   a. \(p \lor q\)
   b. \(p \rightarrow q\)
   c. \(p \land q\)

   \[
   \neg p \\
   \therefore q \\
   \therefore p
   \]

2. Test the validity of the following arguments:
   a. \(p \rightarrow \neg q\)
   b. \(p \rightarrow q\)
   c. \(\neg p \rightarrow \neg q\)

   \[
   \neg p \\
   \therefore q \\
   \therefore \neg p \\
   \therefore q
   \]

3. Test the validity of the following arguments:
   a. \(p \rightarrow q\)
   b. \(p \rightarrow q\)
   c. \(\neg p \land \neg q\)

   \[
   q \rightarrow p \\
   r \rightarrow \neg q \\
   \therefore p \leftrightarrow q \\
   \therefore p \rightarrow \neg r \\
   \therefore r
   \]

4. Write the following arguments in symbolic form and test their validity:
   a. If \(x = 3\), then \(x^2 = 9\)
      \[x^2 \neq 9\]
      Therefore \(x \neq 3\).

   b. If it is raining, I will not play tennis.
      It is not raining.
      Therefore I will play tennis.

   c. All triangles have five sides.
      All five-sided figures are called pentagons.
      Therefore all triangles are pentagons.

   d. Today it is warm and sunny.
      If a day is warm, then it is sunny.
      Therefore today it is warm.

   e. If there is a storm, the streets will be flooded.
      There is not a storm.
      Therefore the streets will not be flooded.

   f. If \(x\) is a chicken, then \(x\) does not have four legs.
      If \(x\) is a dog, then \(x\) has four legs.
      Therefore if \(x\) is a dog, then \(x\) is not a chicken.
g If $x$ is a ewe, then $x$ is a female.
   $x$ is not a female, and so $x$ is not a ewe.

h If Ben smokes or drinks, he will not be fit for the final.
   Ben was fit for the final.
   Therefore he didn't drink.

i If we go to the party we shall leave at 7:00 p.m.
   We shall go to the party or stay home and watch television.
   We didn't leave at 7:00 p.m.
   Therefore we stayed home and watched television.

j If the team plays well, the forwards are outstanding or the defence is superb.
   The defence wasn't superb and the team did not play well.
   Therefore the forwards were outstanding.

k If I join a club, then I will feel secure.
   If I feel secure, then I will go dancing.
   I didn't go dancing.
   Therefore I didn't join a club.

l If I bank my money then I can buy a car.
   If I don't bank my money, then I will be broke.
   Either I will bank my money or I will not buy a car.
   I didn't bank my money.
   Therefore, I am broke.

11.13 Quantifiers

A quantifier is a word or phrase relating to quantity, i.e. how many. Such quantifiers as ‘all’, ‘every’, ‘each’, and ‘any’ have the same meaning and are interchangeable.

The following statements have the same meaning:

- All cats are furry.
- Every cat is furry.
- Any cat is furry.
- Each cat is furry.
- For every cat $x$, $x$ is furry.

The phrase for every $x$ is called a universal quantifier and is written as $\forall x$.

The domain to which $x$ refers, in this case [cats], is often taken for granted. Frequently the domain will be the set of real numbers.

Such quantifiers as ‘some’, ‘at least one’, and ‘there exists’ have the same meaning and are interchangeable.

The following statements have the same meaning:

- Some people are tall.
- At least one person is tall.
- There is a tall person.
- There exists a tall person.
- There exists a person $x$, such that $x$ is tall.

The phrase ‘there exists an $x$ such that’ is called an existential quantifier and is written as $\exists x$. Again, the domain to which $x$ refers is usually understood, although it is sometimes made specific. e.g. $\exists x \in R$ means ‘there exists a real number $x$ such that’.
Example 21

Give translations of the following symbolic forms (real numbers assumed):

a  \( \forall x \exists y (y < x) \)

b  \( \forall a \forall b \exists c (a + b = c) \)

c  \( \forall x \forall y \forall z ((x + y) + z = x + (y + z)) \)

The formulae to which quantifiers apply are included in brackets for clarity.

\[ a \quad \text{For every } x \text{ there exists a } y \text{ such that } y < x. \]

\[ b \quad \text{For every } a, \text{ and for every } b, \text{ there exists a } c \text{ such that } a + b = c, \text{ or } \text{For every } a \text{ and } b \text{ there is a } c \text{ such that } a + b = c, \text{ or } \text{Given any two numbers } a \text{ and } b, \text{ there is a number, } c, \text{ such that } a + b = c. \]

\[ c \quad \text{For every } x, y \text{ and } z, (x + y) + z = x + (y + z). \]

Not all such statements are true. Reversing the quantifiers in part a gives \( \exists y \forall x (y < x) \), i.e. there exists a \( y \) such that, for every \( x, y < x \), which is obviously false. Quantifiers assist in writing statements more precisely, whether we use the verbal or symbolic forms, e.g. The proposition ‘If the sides of a quadrilateral are equal, then its diagonals are perpendicular’ can be more clearly stated using the universal quantifier as follows:

‘For every quadrilateral with equal sides, it is true that the diagonals are perpendicular’. This in turn makes much more obvious what is needed to disprove a proposition.

Consider the negations of propositions involving quantifiers:

<table>
<thead>
<tr>
<th>Proposition, ( p )</th>
<th>Negation, ( \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>All cats are furry</td>
<td>Some cats are not furry</td>
</tr>
<tr>
<td>Some people are tall</td>
<td>All people are not tall, i.e. no people are tall</td>
</tr>
</tbody>
</table>

It would be incorrect and ambiguous to write the negations as:

All cats are not furry
Some people are not tall

The claim: For every \( x \), proposition \( p(x) \) is true, i.e. \( \forall x, p(x) \)

has negation: There exists an \( x \) such that \( p(x) \) is false, i.e. \( \exists x, \text{ such that not } p(x) \) is true

and:

The claim: There exists an \( x \) such that \( q(x) \) is true, i.e. \( \exists x, q(x) \)

has negation: For every \( x \), proposition \( q(x) \) is false, i.e. \( \forall x, \text{ not } q(x) \) is true.

The negations involve exchanging the type of quantifier and reversing the truth value of the proposition.

So, to disprove the above proposition about quadrilaterals, we seek to establish the negation, i.e. ‘there exists a quadrilateral with equal sides whose diagonals are not perpendicular’.

If we write, the proposition ‘All cats are furry’ as ‘For every cat \( x \), \( x \) is furry’, then its negation ‘Some cats are not furry’ can be written as ‘There exists a cat \( x \), such that \( x \) is not furry’.

Similarly, writing the proposition ‘Some people are tall’ as ‘There exists a person \( x \), such that \( x \) is tall’, then its negation can be written as ‘For every person \( x \), \( x \) is not tall.’
Example 22
Write the negation of:

a All cats are furry and some people are tall.
b If all cats are furry, then some people are tall.
c Some people are tall or all cats are furry.
d If some people are not happy, then some people are not healthy.

A

a Since the negation of \( p \land q \) is \( \neg p \lor \neg q \) where \( p \) is the statement ‘All cats are furry’ and \( q \) is the statement ‘Some people are tall’, the negation is:

‘Some cats are not furry or all people are not tall’.

b Since the negation of \( p \rightarrow q \) is \( p \land \neg q \), the required negation is:

‘All cats are furry and all people are not tall’.

c Since the negation of \( p \lor q \) is \( \neg p \land \neg q \), the required negation is:

‘No people are tall and some cats are not furry’.

d Since the negation of \( p \rightarrow q \) is \( p \land \neg q \), the required negation is:

‘Some people are not happy and all people are healthy’.

Exercises 11f

1 Write the negation of each of the following propositions:

a Some numbers are divisible by 5.
b All triangles have three sides.
c Everyone owns a television set.
d At least one house in the street has a front fence.
e Some quadratic equations have two equal solutions.
f There exists a real number \( x \) such that \( 3x = 12 \).
g For every even number \( x \), \( x^2 \) is even.
h No football team wins all games.

2 Write the negation of each of the following compound statements:

a All human beings will eventually die and some quadrilaterals are parallelograms.
b All shoplifters are dishonest and some dishonest people are sent to jail.
c If all cats like milk, then all dogs like meat.
d If some people are wealthy, some people are poor.
e Some numbers are even or some numbers are prime.
f Some triangles are right-angled and some quadrilaterals are squares.
g If all dogs like meat, then at least one cat likes milk.
h There exists a person who is 200 cm tall and whose hair is black.

3 Translate the following into reasonable English:

a \( \exists x \forall y, y \text{ adores } x \)
b \( \exists x \forall y, x \text{ adores } y \)
c \( \forall y \exists x, y \text{ adores } x \)
d \( \forall a \forall b \exists c, a + c = b \)
e \( \exists p \exists q \forall r, p + r = qr \)
4 Determine the truth or falsity of the following statements about real numbers:
   a) \( \exists x \exists y \forall z, z = x + y \)
   b) \( \forall x \forall y \exists z, x = \frac{y}{z} \)
   c) \( \exists a \forall b \forall c, a = \log bc \)
   d) \( \forall a \exists b \forall c, b = a^2 + c^2 \)
   e) \( \forall a \exists b \forall c, c < a + b \) implies \( c < b \)

5 Write contractions for each of the following statements about real numbers:
   a) \( \forall p, z = p^2 \)
   b) \( \forall a \forall b, x < a^2 + b^2 \)
   c) \( \exists a \exists b, x = a^2 + b^2 \)
   d) \( \forall x \exists y, |f(x)| < y \)